CONSTRUCTIVIZABILITY CRITERION FOR AN ABELIAN GROUP

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We establish a criterion determining whether an Abelian group is constructivizable, which is then used to prove that for any natural number $r \geq 1$, there exists a principal computable enumeration $\gamma_r$ of the class $K_r$ of all constructive groups whose torsion-free ranks are distinct from zero and do not exceed $r$.

INTRODUCTION

The notion of a principal enumeration plays an important part in algorithm theory. Basic results concerning the existence of principal computable enumerations for distinct classes of recursively enumerable sets, partial recursive functions, constructive algebraic systems, etc. are given in [1, 2]. Dobritsa in [3, Question 72] posed the following problem.

Denote by $I$ a set of natural numbers not containing 0 and having at least 2 elements. Can, then, a computable class of constructive Abelian groups with Prüfer rank in $I$ have a principal computable indexation, that is, one to which all computable indexations of that class are reduced?

By a Prüfer rank, or torsion-free rank, of a group $A$ we mean the rank of a quotient group $A/T$ w.r.t. a periodic part $T$. Here are some well-known results related to computable constructive classes of Abelian groups:

1. The following classes have principal computable enumerations: (a) a class of periodic groups (cf. [4]), and (b) a class of groups whose torsion-free ranks are equal to $n$ for any $n \in \omega$ (cf. [5]).

2. The following classes are computable but have no principal computable enumerations (cf. [5]): (a) a class of groups whose torsion-free ranks do not exceed $n$ for any $n \in \omega$, and (b) a class of all groups whose torsion-free ranks are finite.

In this article, we reason to answer in the affirmative the above-cited question for the case where $I = \{1, 2, ..., r\}$, where $r > 1$ is an arbitrary natural number. To do this, use will be made of the constructivizability criterion for an Abelian group (Thm. 1), which generalizes a similar criterion of being constructivizable for a torsion-free Abelian group from [6] and is the basic result of the present article.

For all the concepts used undefined in the argument, we ask the reader to consult [1, 2, 7, 8]. We bring out some of them. In what follows, by a group we mean a not more than countable Abelian group. The set of all natural numbers is denoted by $\omega$, and the set of all primes — by $P; p_0, p_1, ...$ is a sequence of all primes. Let $A$ be a group. A system of elements $a_1, ..., a_n$ in $A$ is called $t$-dependent, $t \in \omega$, if there exist integers $m_1, ..., m_n$ (not all of whom are equal to zero) such that $|m_t| \leq t$ and the equality $m_1 a_1 + ... + m_n a_n = 0$ holds in $A$. Otherwise, we call the system $t$-independent. If $M \subseteq A$, by $\text{gr}(M)$ we denote a subgroup of $A$ generated by the set $M$. Let $[m_0, ..., m_n]_\omega ([m_0, ..., m_{n-1}]_n)$ be a number of the sequence $m_0, ..., m_n$ in the numbering of all finite sequences ($n$-tuples) of natural numbers. On the sets $\mathbb{Z}^k$ and $\omega^k$, the usual lexicographic order is defined.
1. CONSTRUCTION OF $A[(T, \mu), f, g]$

Assume that a constructive periodic group $(T, \mu)$ and partial recursive functions $f(k, s, n)$ and $g(s, t)$ are given. Using these functions, we proceed by steps over $t$ to construct a group $A[(T, \mu), f, g] = A$ by specifying its generators and defining relations. The periodic part of $A$ is the group $T$. Let the function $g(s, t)$ be total and $s_0$ be a least number such that either $\lim_{t \to \infty} g(s_0, t)$ does not exist, or $\lim_{t \to \infty} g(s_0, t) = 0$. If there is no such number, put $s_0 = \omega$. The rank $r$ of the quotient group $A/T$ is equal to $\max \{s_0, 1\}$.

Suppose that a group $G$ is given, and the rank of the quotient $G/T$ of $G$ w.r.t. its periodic part is equal to $r \in \omega \cup \{\omega\}$. Following [9], we introduce the concept of a $p$-primitive subgroup $\Gamma_p$ of the group $G$. For simplicity, assume $r < \omega$. In $G/T$, choose a maximal linearly independent system of elements $a_0, \ldots, a_{r-1}$. A subgroup $\Gamma_p$ consisting of elements $g$ such that there exist integers $n, m_1, \ldots, m_k, k < r$, for which

$$p^n g = m_1 a_1 + \ldots + m_k a_k \pmod{H},$$

is then called a $p$-primitive subgroup $\Gamma_p$ of $G$. It is easy to see that $G$ is generated by subgroups $\Gamma_p$, where $p \in P$.

In order to define $A$, it suffices to construct its basis $a_0, \ldots, a_{r-1}$ and $p$-primitive subgroups $\Gamma_p, p \in P$. Let $k$ be a fixed natural number. We construct $\Gamma = \Gamma_p$ by steps over $t$. Denote by $\Gamma^t$ a partial subgroup that obtains at the end of step $t$. Simultaneously, a basis for $A/T$ will be constructed. The construction of the latter, we note, does not depend on the number $p_k$ but depends on the function $g$. The periodic part of $\Gamma$ is the group $T$. The group $\Gamma$ is constructed by specifying its generators and defining relations. Assume that elements of $T$ are all natural numbers, which we denote by $\alpha, \beta, \gamma, a_1, \ldots$, and its zero element is 0.

At step $t$, we construct the basis $b^t_0, \ldots, b^t_{m^t-1}$.

This system of elements is assumed $t$-independent at step $t$. It may become dependent at succeeding steps. The numbers $m^t, N^t, n^t_i$ and elements $b^t_j, i < m^t, j \leq n^t_i$, are also defined at step $t$. If $j = 0$, then always $b^{t+1}_0 = b^t_{s+1}, s \neq k$. Below we drop $k$ from the above-mentioned symbols. The elements $b^t_1$ and $b^t_0$ are denoted by $c^t_1$ and $c^t_0$, respectively. The universe of a partial group $\Gamma^t$ is

$$\{\alpha + z_0 c^t_0 + \ldots + z_{m^t-1} c^t_{m^t-1} \mid \alpha \in T, |z_i| < N^t\}.$$

We make the convention that if the value of some parameter distinct from $N^t$ is not defined explicitly at step $t$, then it should be assumed equal to its value at a preceding step, and we also put $N^t = \max \{N^{t-1}, t\}$. If, however, nor is that value defined at the preceding step, we set it equal to zero at step $t$.

Step 0. Introduce the symbol $b_0$ and put $N^0 = m^0 = 1, n^0 = 0$. Assume that the relation $b^t_0 = b_0$ is satisfied for all $t$.

Step $t + 1$. Let $t + 1 = [s, l]_\omega$. Compute $g(s, l)$. If it is undefined, then

$$\Gamma = T \oplus (c^t_0) \oplus \ldots \oplus (c^t_{m^t-1}).$$

(1)

Let $g(s, l)$ be defined. If $s > m^t$, we pass to the next step. There are two cases to consider:

1. Let $g(s, l) \neq g(s, l - 1)$. If $s = 0$, we pass to the next step. Let $s > 0$. If $s = m^t$ and $g(s, l) \neq 0$, we put $m^{t+1} = m^t + 1$. But if $s = m^t$ and $g(s, l) = 0$, then we pass to the next step.

2. Let $s < m^t$. Here, the inequality $g(s, l) \neq g(s, l - 1)$ means that elements $b_0^t, \ldots, b^t_s$ are $t$-dependent and $b^t_s, b^t_{s+1}$ are $t$-independent. Denote by $\Gamma^s$ a partial group on the set

$$\{\alpha + z_0 c^t_0 + z_1 c^t_1 + \ldots + z_{s-1} c^t_{s-1} \mid |z_0| < p^t_0, |z_1|, \ldots, |z_{s-1}| \leq N^t\}.{$$