1. Introduction

Convexity is an important mathematical concept. This is justified, e.g., by the fact that in a number of classical problems of calculus and topology, we see that the following case is typical: a problem that is purely topological in its statement has a purely topological solution in the case of finite-dimensional spaces, and in the general case, it is solvable only under an additional not topological condition, the convexity. For a detailed survey of the basic directions of multivalued calculus, see [1, 2]. Here we restrict ourselves to several classical problems.

(a) Extension of continuous mappings. If a space $Y$ is metrizable, $n$-connected ($Y \in C^n$), and locally $n$-connected ($Y \in LC^n$), then every continuous mapping $f : A \to Y$ of a closed subset $A$ of a metric space $X$ with Lebesgue dimension $\dim X \leq n + 1$ is continuously extended to the whole space $X$. This is the classical Kuratowski–Dugundji continuation theorem [3]. For a completely metrizable space $Y$, this theorem also holds for a paracompact domain of $X$; see [33]. In a slightly different terminology, the condition $Y \in (LC^n \wedge C^n)$ is equivalent to the fact that $Y$ is an absolute extensor for the class of $(n + 1)$-dimensional metric spaces: $Y \in AE(n + 1)$. At the same time, there exist infinite-dimensional compacta $Y$ that are $n$-connected and locally $n$-connected for all natural $n$, but which, however, are not extensors for the class of all metric spaces; see [3]. For the latter class, the extension problem is positively solved in the case of a convex $Y$. The classical Borsuk–Dugundji theorem [3] says that every convex subset $Y$ of a locally convex topological vector space is an absolute extensor: $Y \in AE$.

(b) Continuous selections of multivalued mappings. A single-valued mapping $f : X \to Y$ is called a selection of a multivalued mapping $F : X \to 2^Y$, where $2^Y$ is the set of all nonempty subsets of $Y$, if, for all $x \in X$, the point $f(x)$ lies in the set $F(x)$. The classical finite-dimensional Michael selection theorem [33] can be reformulated (see [37]) in the form of equivalence of the following two conditions for any completely metrizable space $Y$ and any family $A \subseteq 2^Y$ that is saturated, i.e., $(L \in A) \wedge (y \in L) \implies \{y\} \in A$:

1. any lower semicontinuous mapping $F : X \to A$ of any paracompact space $X$ of dimension $\leq n + 1$ has a continuous single-valued selection;

2. all elements of the family $A$ are $n$-connected and any lower semicontinuous mapping $F : X \to A$ of any paracompact space $X$ of dimension $\leq n + 1$ has local continuous selections, i.e., any partial selection of the mapping $F$ is extended from its closed domain to a certain open neighborhood of the domain.

At the same time, examples by Pixley [40] and Michael [36] show that there is no class $\Gamma$ of topological spaces such that (for such $Y$ and $A$) the following two conditions are equivalent:

3. any lower semicontinuous mapping $F : X \to \Lambda$ of any paracompact space $X$ has a single-valued continuous selection;

4. all elements of the family $\Lambda$ lie in $\Gamma$ and any lower semicontinuous mapping $F : X \to \Lambda$ of any paracompact space $X$ has local continuous selections.

Thus, for arbitrary paracompact domains of lower semicontinuous multivalued mappings, there is no purely topological analog of solution of the finite-dimensional selection problem. As in case (a), the convexity

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comes in the play. A lower semicontinuous closed-valued and convex-valued multivalued mapping of a paracompact space into a Banach space has a continuous selection; probably, this is the most well known and widely used Michael selection theorem; see [32].

(c) Continuous (graphical) approximations. A single-valued mapping \( f : X \to Y \) of metric spaces is called an \( \varepsilon \)-approximation of a multivalued mapping \( F : X \to 2^Y \) if the graph of the mapping \( f \) lies in the \((\varepsilon \times \varepsilon)\)-neighborhood of the graph of the mapping \( F \). The approximability (i.e., the existence of \( \varepsilon \)-approximations for all \( \varepsilon > 0 \)) of a closed-valued mapping of an AE-compactum \( X \) into itself guarantees the existence of a fixed point for this mapping. Namely in this way, the fixed-point theorems by Kakutani, Bonnenblast–Karlin, and Gliksberg [15, 21, 26] were proved for mappings of convex compacta. In the case of an \( n \)-dimensional (not necessarily compact) \( X \), the approximability of an upper semicontinuous compact-valued mapping \( F : X \to Y \) is a consequence of the purely topological, so-called \( UV^n \)-properties of values of \( F(x) \), that is, for any neighborhood \( U \) of the set \( F(x) \), there is a smaller neighborhood \( V \) for which the identical embedding \( V \subseteq U \) is homotopy trivial in dimension \( n \). In a very general form, this was proved recently by Shchepin and Brodskii [10] by using the ideas of [9, 30]. For an infinite-dimensional compactum \( X \), the approximability of the mapping \( F : X \to X \) is guaranteed by \( UV(\infty) \)-properties of values of this mapping; this result was obtained by Granas, Gorniewicz, and Kryszewsky [22] and Gutev [24]. The statement converse to this approximation theorem in the case where \( X \) is locally finite-dimensionally polyhedral was obtained by Kryszewsky in [29]. For an arbitrary domain of \( X \) and for a compact-valued upper semicontinuous mapping \( F \), in both cases (a) and (b), a purely topological solution of the approximation problem is not known. The additional condition that the mapping \( F \) is convex-valued immediately yields the solution of this problem even in the case where it is not assumed that it is convex-valued; this is the classical von Neumann–Celina approximation theorem; see [11].

(d) Minimax theorems. The classical von Neumann–Sion minimax theorem (see [51]) for a real-valued function \( f : X \times Y \to \mathbb{R} \) asserts that the relation

\[
\min_Y \max_X f(x, y) = \max_X \min_Y f(x, y)
\]

holds under the following conditions:

(5) \( X \) and \( Y \) are convex compacta;

(6) \( f(x, y) \) is upper semicontinuous and \( f(x, \cdot) \) is lower semicontinuous for all \( (x, y) \in X \times Y \);

(7) the sets \( \{x \in X \mid f(x, y_0) \geq c\} \) and \( \{y \in Y \mid f(x_0, y) \leq c\} \) are convex for all \( c \in \mathbb{R}, (x_0, y_0) \in X \times Y \).

There are a large number of papers where an analog of this theorem is proved by considering one or another convexity in (5) and (7), which is axiomatically given in [25, 31, 52, 53, 54]. It is not possible to directly assert that one cannot avoid the convexity in this theorem. For example, the methods going back to the paper [55] by Wu allowed one to obtain topological minimax theorems [20, 28, 48], in which, however, an essential "convex" aspect remains: one needs the centered connectivity of the family of sub- and superlevels of a function \( f \), that is, the finite intersections

\[
\bigcap \{\{x \in X \mid f(x, y_i) \geq c\} \mid y_1, y_2, ..., y_n \in Y\}
\]

of superlevel sets are assumed to be empty or connected in these papers. In all the cases, one uses assertions of the type of the KKM principle [18, 31], in which the convexity is again essential.

Summarizing what was said above, we have to stress the fact that a purely topological analog of the convexity (in the infinite-dimensional case) is not known, and it does not exist at all from the viewpoint of the results of (b). Therefore, one naturally tries to pass from topological terms to more restrictive metrical ones, and in this framework, tries to answer the question: to what extent can one neglect the convexity in the classical theorems of multivalued calculus which are listed above? To make the last question more precise, we have to formalize the concept of "neglecting the convexity." Thus, we arrive at the problem of introducing a certain nonconvexity characteristic of subsets of a Banach space. In this case, an object that characterizes the nonconvexity of sets should satisfy at least two requirements. First, the nondegeneracy of such an object

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