UNIQUENESS OF THE SOLUTION OF THE FIRST MIXED PROBLEM FOR A TWO-DIMENSIONAL NONLINEAR PARABOLIC EQUATION

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UDC 514.946

The first mixed problem for a two-dimensional nonlinear parabolic equation with nonlinear occurrences of the second derivatives of the unknown function is considered. Under the assumption that a solution possessing continuous second derivatives with respect to the coordinate variables exists in a closed cylinder and under certain constraints on the initial data of the problem, the uniqueness of this solution is proved by applying the longitudinal version of the method of straight lines. Bibliography: 4 titles.

1. Consider the parabolic equation

\[ u_t = f(t, x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, u_{xxy}) \]  

in the cylinder \( Q_T = \{ (x, y) \in \Pi, 0 \leq t \leq T \} \), where \( \Pi \) is a rectangular region \( [x_1 \leq x \leq x_2, y_1 \leq y \leq y_2] \), with the initial condition

\[ u(0, x, y) = \Phi(x, y), (x, y) \in \Pi, \]

and the boundary conditions

\[ u(t, x_1, y) = \varphi_1(t, y), \quad u(t, x_2, y) = \varphi_2(t, y), \quad y_1 \leq y \leq y_2, \]

\[ u(t, x, y_1) = \varphi_3(t, x), \quad u(t, x, y_2) = \varphi_4(t, x), \quad 0 \leq t \leq T, \]

on \( \Gamma_T = \{ x \in \partial \Pi, 0 \leq t \leq T \} \). We assume that the function \( f \) is continuous with respect to all of its arguments and has continuous partial derivatives with respect to \( u, u_x, u_y, u_{xx}, u_{yy}, u_{xy} \) for \( (t, x, y) \in Q_T \) and \(-\infty < u, u_x, u_y, u_{xx}, u_{yy}, u_{xy} < \infty \); the functions \( \varphi_k, k = 1, 2, 3, 4 \), continuously depend on their arguments and are continuously differentiable with respect to \( t; \varphi_1(t, y) \) and \( \varphi_2(t, y) \) are twice continuously differentiable with respect to \( y \), whereas \( \varphi_3(t, x) \) and \( \varphi_4(t, x) \) are twice continuously differentiable with respect to \( x \).

We also assume that the problem \((1) \text{-} (3)\) has a solution satisfying the condition

\[ u(t, x, y) \in C^{1,2}(Q_T). \]  

We construct a scheme of the method of straight lines. To this end, we leave the variable \( t \) continuous and discretize the variables \( x \) and \( y \). We denote \( x_i = x_1 + ih, i = 1, 2, \ldots, n_1; y_j = y_1 + jh, j = 1, 2, \ldots, n_2 \). In what follows, \( X_k \equiv x_{k-1} \frac{h}{n_1}, Y_k \equiv y_{k-1} \frac{h}{n_2} \); \( U(t) = \{ u(t, X_k, Y_k) \}, U^r(t) = \{ u_x(t, X_k, Y_k) \}, U^y(t) = \{ u_y(t, X_k, Y_k) \}, U^{xx}(t) = \{ u_{xx}(t, X_k, Y_k) \}, \) \( U^{yy}(t) = \{ u_{yy}(t, X_k, Y_k) \}, \) and \( U^{xy}(t) = \{ u_{xy}(t, X_k, Y_k) \} \) are vector functions of dimension \( n_1 \times n_2 \);
I and \([0]\), respectively. At the mesh points \(x_i, y_j \quad i = 1, 2, \ldots, n_1, j = 1, 2, \ldots, n_2\), we replace the derivatives occurring in Eq. (1) by the expressions

\[
U_x(t) \sim \frac{1}{2h} B_{[0]}^{n_2} B_{10}\{0\} U(t) + \frac{1}{2h} R_1(t),
\]

\[
U_y(t) \sim \frac{1}{2h} B_{[0]}^{n_2} B_{01}\{0\} U(t) + \frac{1}{2h} R_2(t),
\]

\[
U_{xx}(t) \sim \frac{1}{h^2} B_{[0]}^{n_2} B_{11}\{0\} U(t) + \frac{1}{h^2} R_3(t),
\]

\[
U_{yy}(t) \sim \frac{1}{h^2} B_{[0]}^{n_2} B_{22}\{0\} U(t) + \frac{1}{h^2} R_4(t),
\]

\[
U_{xy}(t) \sim \frac{1}{h^2} B_{[0]}^{n_2} B_{12}\{0\} U(t) + \frac{1}{h^2} R_5(t) + \frac{1}{2} R_{xy}(t),
\]

where

\[
R_1(t) = (-\varphi_1(t, y_1), 0, \ldots, 0, \varphi_2(t, y_1), -\varphi_1(t, y_2), 0, \ldots, 0, \varphi_2(t, y_2), \ldots, -\varphi_1(t, y_{n_2}), 0, \ldots, 0, \varphi_2(t, y_{n_2}))^T,
\]

\[
R_2(t) = (-\varphi_3(t, x_1), -\varphi_3(t, x_2), \ldots, -\varphi_3(t, x_{n_1}), 0, 0, \ldots, 0, \varphi_4(t, x_1), \varphi_4(t, x_2), \ldots, \varphi_4(t, x_{n_1}))^T,
\]

\[
R_3(t) = (\varphi_1(t, y_1), 0, \ldots, 0, \varphi_2(t, y_1), \varphi_1(t, y_2), 0, \ldots, 0, \varphi_2(t, y_2), \ldots, \varphi_1(t, y_{n_2}), 0, \ldots, 0, \varphi_2(t, y_{n_2}))^T,
\]

\[
R_4(t) = (\varphi_3(t, x_1), \varphi_3(t, x_2), \ldots, \varphi_3(t, x_{n_1}), 0, 0, \ldots, 0, 0, 0, \ldots, 0, \varphi_4(t, x_1), \varphi_4(t, x_2), \ldots, \varphi_4(t, x_{n_1}))^T,
\]

\[
R_5(t) = (-\varphi_3(t, x_1), -\varphi_3(t, x_2), \ldots, -\varphi_3(t, x_{n_1}), 0, 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0)^T.
\]

\[
R_{\varphi}(t) = \begin{pmatrix}
-\frac{\partial \varphi_1}{\partial y}(t, y_1), 0, 0, \ldots, 0, \frac{\partial \varphi_1}{\partial y}(t, y_2), 0, 0, \ldots, 0, \ldots, -\frac{\partial \varphi_1}{\partial y}(t, y_{n_2}), 0, 0, \ldots, 0
\end{pmatrix}^T,
\]

\[
R_{\varphi\varphi}(t) = \begin{pmatrix}
\frac{\partial^2 \varphi_1}{\partial y^2}(t, y_1), 0, 0, \ldots, 0, \frac{\partial^2 \varphi_1}{\partial y^2}(t, y_2), 0, 0, \ldots, 0, \ldots, \frac{\partial^2 \varphi_1}{\partial y^2}(t, y_{n_2}), 0, 0, \ldots, 0
\end{pmatrix}^T,
\]

are vector functions of dimension \(n_1 \times n_2\), which, in view of (4) (see [1]), converge, as \(h \to 0\), to the values of the vector functions \(U_x(t), U_y(t), U_{xx}(t), U_{yy}(t), \) and \(U_{xy}(t)\); respectively.

With the problem (1)–(3) we associate the following Cauchy problem for a system of ordinary differential equations:

\[
\frac{dv_i}{dt} = f(t, X_i, Y_i, v_i, v_i^x(V), v_i^y(V), v_i^{xx}(V), v_i^{yy}(V), v_i^{xy}(V)),
\]