SOLVABILITY OF A PERIODIC BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH ($\beta, \gamma, \delta$)-COMPARISON PAIRS

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Existence theorems are proved for an incomplete set of upper and lower functions. Bibliography: 1 title.

Let $\beta, \gamma, \delta \in \mathbb{R}^n$ be vectors with components $\beta_j \in \{-1,1\}$, $\gamma_j \in \{0,1\}$, and $\delta_j \in \{0,1\}$ ($j = 1,\ldots,n$), respectively. Consider the periodic boundary-value problem for a system of first-order ordinary differential equations

$$\frac{du_i}{dt} = f_i(t, u_1, \ldots, u_n), \quad 0 \leq t \leq 1 \quad (i = 1, \ldots, n)$$

with boundary conditions

$$u_i(0) = u_i(1) \quad (i = 1, \ldots, n).$$

For this system, a pair of vectors $v \leq w, v, w \in \mathbb{R}^n$, such that $\gamma_j u_j \geq v_j, \delta_j u_j \leq w_j$ ($j = 1,\ldots,n$), and, for $0 \leq t \leq 1$ and $i = 1,\ldots,n$,

$$\beta_i \gamma_i \left[ f_i(t, u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_n) - \frac{du_i}{dt} \right] = \beta_i \gamma_i f_i(t, u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_n) \leq 0,$$

$$\beta_i \delta_i \left[ f_i(t, u_1, \ldots, u_{i-1}, w_i, u_{i+1}, \ldots, u_n) - \frac{dw_i}{dt} \right] = \beta_i \delta_i f_i(t, u_1, \ldots, u_{i-1}, w_i, u_{i+1}, \ldots, u_n) \geq 0$$

is said to be a ($\beta, \gamma, \delta$)-comparison pair. The existence of a nontrivial comparison pair permits one to weaken conditions in existence theorems. In proving this claim, the fixed-point theorem provided below will be used.

Consider the periodic boundary-value problem for a system of first-order ordinary differential equations

$$\frac{du_i}{dt} = f_i(t, u_1, \ldots, u_n), \quad 0 \leq t \leq 1 \quad (i = 1, 2, \ldots, n),$$

$$u_i(0) = u_i(1) \quad (i = 1, \ldots, n).$$

We assume that

(1) $\alpha \in \mathbb{R}^n$ is a vector with components $\alpha_j \in \{-1,0,1\}$ ($j = 1,\ldots,n$);

(2) $\beta \in \mathbb{R}^n$ is a vector with components $\beta_j \in \{-1,1\}$ ($j = 1,\ldots,n$);

(3) $\gamma \in \mathbb{R}^n$ is a vector with components $\gamma_j \in \{0,1\}$ ($j = 1,\ldots,n$);

(4) $\delta \in \mathbb{R}^n$ is a vector with components $\delta_j \in \{0,1\}$ ($j = 1,\ldots,n$);

(5) $m, M \in \mathbb{R}^n$ are vectors with components $m_j \leq 0$ and $M_j \geq 0$ ($j = 1,\ldots,n$), respectively.

**Theorem 1.** Let

$$\beta_i f_i(t, u_1, \ldots, u_n) \text{sign} u_i \geq a_i(t, u_1, \ldots, u_{i-1}) |u_i| - g_i(t, u_1, \ldots, u_{i-1}),$$

$$\int_0^1 a_i(t, u_1, \ldots, u_{i-1}) dt > 0, \quad g_i(t, u_1, \ldots, u_{i-1}) \geq 0.$$
\[
\beta_i \gamma_i f_i(t, u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) \leq 0, \tag{4}
\]
\[
\beta_i \delta_i f_i(t, u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) \geq 0, \tag{5}
\]
and let the functions \( f_i(t, u_1, \ldots, u_n) \), \( g_i(t, u_1, \ldots, u_{i-1}) \), and \( a_i(t, u_1, \ldots, u_{i-1}) \) \((i = 1, \ldots, n)\) be continuous for \( 0 \leq t \leq 1 \), \( \gamma_j u_j(t) \geq 0 \), and \( \delta_j u_j(t) \leq 0 \) \((j = 1, \ldots, n)\).

Then the periodic boundary-value problem (1), (2) has at least one solution \( u(t) \) such that, for \( 0 \leq t \leq 1 \),

\[
\gamma_j u_j(t) \geq 0, \quad \delta_j u_j(t) \leq 0 \quad (j = 1, \ldots, n). \]

**Proof.** Along with the problem (1), (2), we consider the \( \lambda \)-family \((0 \leq \lambda \leq 1)\) of periodic boundary-value problems

\[
\beta_i \frac{du_i}{dt} - a_i(t, \langle u_1, \ldots, u_{i-1} \rangle)u_i = \lambda \left[ \beta_i f_i(t, \langle u_1, \ldots, u_{i-1} \rangle) - a_i(t, \langle u_1, \ldots, u_{i-1} \rangle) \langle u_i \rangle \right] (i = 1, \ldots, n), \tag{6}
\]
\[
u_i(0) = u_i(1) \quad (i = 1, \ldots, n). \tag{7}
\]

Here,

\[
\langle u_j \rangle = \begin{cases} 0 & \text{if } \gamma_j = \delta_j = 1, \\ \max \{0, u_j\} & \text{if } \gamma_j = 1, \delta_j = 0, \\ \min \{0, u_j\} & \text{if } \gamma_j = 0, \delta_j = 1, \\ u_j & \text{if } \gamma_j = \delta_j = 0. \end{cases}
\]

We claim that, for \( 0 \leq t \leq 1 \) and \( 0 \leq \lambda \leq 1 \), any solution \( u(t, \lambda) = \{u_1(t, \lambda), \ldots, u_n(t, \lambda)\} \) of the problem (6), (7) satisfies the inequalities \( \gamma_j u_j(t, \lambda) \geq 0 \) and \( \delta_j u_j(t, \lambda) \leq 0 \) \((j = 1, \ldots, n)\).

We will establish the inequalities \( \gamma_j u_j(t, \lambda) \geq 0 \) \((j = 1, \ldots, n)\). Suppose the contrary, i.e., let, for some \( i \) and \( \lambda \), \( \min_{0 \leq t \leq 1} u_i(t, \lambda) = u_i(t^*, \lambda) < 0 \). Then the following two cases are possible:

(1) \( u_i(t, \lambda) \leq 0 \) for all \( 0 \leq t \leq 1 \),

(2) there exist \( t_1, t_2 \in [0, 1] \) such that \( u_i(t_1, \lambda) = u_i(t_2, \lambda) = 0 \).

In the first case, we have \( \langle u_i(t, \lambda) \rangle = 0 \) for all \( 0 \leq t \leq 1 \), whence

\[
\beta_i \frac{du_i}{dt} \bigg|_{t=1} \geq 0, \quad \frac{du_i}{dt} \bigg|_{t=0} < 0.
\]

In the second case, for \( t = t_1 \), from Eqs. (6) and inequalities (4) we have

\[
\beta_i \frac{du_i}{dt} \bigg|_{t=t_1} \leq 0,
\]

which implies that \( u_i(t, \lambda) \geq 0 \). Thus, \( u_i(t, \lambda) \equiv 0 \), which contradicts the inequality \( u_i(t^*, \lambda) < 0 \). In the second case, for \( t = t_1 \), from Eqs. (6) and inequalities (4) we have

\[
\beta_i \frac{du_i}{dt} \bigg|_{t=t_1} \geq 0,
\]

which leads to a contradiction if \( \beta_i = 1 \). For \( t = t_2 \), from Eqs. (6) and inequalities (4) it follows that

\[
\beta_i \frac{du_i}{dt} \bigg|_{t=t_2} \leq 0,
\]

whence, for \( \beta_i = -1 \), we have

\[
\frac{du_i}{dt} \bigg|_{t=t_2} \geq 0,
\]

3366