REGULARITY THEORY FOR THE \((m,l)\)-LAPLACIAN PARABOLIC EQUATION

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We present results on regularity for generalized solutions of equations of the form

\[ u_t - \text{div}(|u|^l \nabla u)^{m-1} \nabla u = 0, \quad m > 1, \quad l > 1 - m, \]

obtained recently by the author. We prove a local \(L_\infty\) estimate for generalized solutions of this equation under the following condition on the parameters \(m, l\):

\[ \frac{\sigma + 1}{\sigma + 2} > \frac{1}{m} - \frac{1}{n}, \quad \sigma = \frac{l}{m-1}, \quad m > 1, \quad l > 1 - m. \]

This condition was found by the author in a previous paper. It was shown there that this condition is necessary for local boundedness of a generalized solution. Bibliography: 18 titles.

Dedicated to the memory of A. P. Oskolkov

1. INTRODUCTION

Consider the \((m,l)\)-Laplacian parabolic equation

\[ \mathcal{F}[u] := u_t - \text{div}(|u|^l \nabla u)^{m-2} \nabla u = 0, \quad m > 1, \quad l > 1 - m, \tag{1.1} \]

where \(\nabla u = \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)\) is the spatial gradient of a solution \(u\). Similar equations arise in the study of turbulent filtration of a gas or a fluid through porous media, in theoretical glaciology, and in the theory of non-Newtonian fluids.

Equation (1.1) is the main model for the so-called doubly nonlinear parabolic equations (DNPE). The existence of a generalized solution for the Cauchy–Dirichlet problem for Eq. (1.1) was first proved by Raviart and J.-L. Lions at the end of the 60s. The regularity problem for DNPE has been investigated only recently.

In this paper, we present the recent theory for DNPE of arbitrary dimension \((n \geq 1)\).

We say that Eq. (1.1) (and a similar DNPE) is an equation of the type of slow, normal, or fast diffusion if \(m + l > 2\), \(m + l = 2\), or \(m + l < 2\), respectively.

In this paper, we always consider DNPE in the cylinder \(Q_T = \Omega \times (0,T)\), where \(\Omega\) is a bounded open set in \(\mathbb{R}^n\), \(n \geq 1\).

Consider the full set of admissible parameters \(m, l\):

\[ \mathcal{D} := \{m > 1, l > 1 - m\} \subset \mathbb{R}^2. \]

Obviously, we have

\[ \mathcal{D} = S \cup N \cup \mathcal{F}, \]

where \(S\), \(N\), and \(\mathcal{F}\) are the subsets of \(\mathcal{D}\) corresponding to equations of slow, normal, and fast diffusion type, respectively.

In the one-dimensional case \((n = 1)\), some regularity results for generalized solutions of the Cauchy and Cauchy–Dirichlet problems for the equation

\[ u_t = \frac{d}{dx}(|u|^l |u_x|^{m-2} u_x), \quad m + l > 2, \]
were obtained in [1, 2] (see also the references therein).

The regularity results for DNPE of arbitrary dimension presented in this paper are based mainly on the results of [3-13]. Note that the $L_\infty$ and Hölder estimates established in these papers can be considered as an appropriate development of well-known results for quasilinear parabolic and degenerate parabolic equations obtained by Ladyzhenskaya-Uraltseva [14], DiBenedetto, and DiBenedetto-Chen [15].

At the end of this paper, we prove a local $L_\infty$-estimate for generalized solutions of Eq. (1.1) under the exact condition on the parameters $m, l$ first discovered in [3].

2. $L_\infty$-ESTIMATES

Definition 2.1. A function $u$ is called a generalized solution of Eq. (1.1) if

(a) $u \geq 0$ in $Q_T$, $u \in C([0, T]; L^{\sigma+2}(\Omega))$, and $\nabla u^{\sigma+1} \in L_m(Q_T; \mathbb{R}^n)$, where $\sigma = \frac{l}{m-1}$;
(b) $u$ satisfies (1.1) in the sense of distributions, i.e.,

$$
\int_0^T \int_{Q_T} (u \Phi_t - \nabla u^{\sigma+1} \cdot \nabla \Phi) \, dx \, dt = 0
$$

for any $\Phi \in W^{1}_{m}(Q_T) \cap L_\infty(Q_T),

where $Q_{t_1,t_2} = \Omega \times [t_1, t_2]$.

The problem of establishing local $L_\infty$-estimates for generalized solutions of Eq. (1.1) uncovers [3] a pathological set $\omega$ which can be defined by the relation

$$
\mathcal{D} \setminus \omega := \{ (m, l) \in \mathcal{D} : \hat{W}^{1}_{m}(\Omega) \text{ is compactly imbedded in } L_{\frac{\sigma+2}{\sigma+1}}(\Omega) \},
$$

where $\sigma = \frac{l}{m-1}$.

The following statements are obvious:

1. the analytical version of the definition of the set $\omega$ has the form

$$
\omega = \left\{ (m, l) \in \mathcal{D} : \frac{\sigma+1}{\sigma+2} \leq \frac{1}{m} - \frac{1}{n}, \sigma = \frac{l}{m-1} \right\};
$$

2. $\omega \subset \mathcal{F};$
3. $\omega = \emptyset$ in the case $n = 1$.

Theorem 2.1. If $(m, l) \in \mathcal{D} \setminus \omega$, then any generalized solution $u$ of Eq. (1.1) is locally bounded. In addition, for any $Q' = \Omega' \times [\varepsilon, T], \tilde{Q}' \subset \Omega$, and $\varepsilon > 0$ we have

$$
\sup_{Q'}(u, Q') \leq c
$$

with some constant $c$ depending only on $n, m, l, \text{dist}(\Omega', \partial \Omega), \varepsilon, \sup_{t \in [0, T]} \int_{\tilde{Q}'} |u^{\sigma+2}| \, dx$, and $\sup_{t \in [0, T]} \int_{\tilde{Q}'} |\nabla u^{\sigma+1}|^m \, dx \, dt$,

where $\sigma = \frac{l}{m-1}$.

If $(m, l) \in \omega$, then estimate (2.1) does not hold.

Theorem 2.1 was stated in [3]. The impossibility of estimate (2.1) in the case $(m, l) \in \omega$ was proved there. However, instead of proving a local $L_\infty$-estimate in the case $(m, l) \in \mathcal{D} \setminus \omega$, we preferred to prove in [3] a global version of the maximum modulus estimate for some regularizations of DNPE, including Eq. (1.1) with $(m, l) \in \mathcal{D} \setminus \omega$. We show in Sec. 9 that a proof of a local $L_\infty$-estimate in the case $(m, l) \in \mathcal{D} \setminus \omega$ can be based on the arguments given in [3].

Remark 2.1. The substance of Theorem 2.1 is a local $L_\infty$-estimate for solutions of Eq. (1.1) for the best possible condition on the parameters $m, l$. Local estimates for the maximum modulus of solutions for Eq. (1.1) for other (not necessarily best possible) conditions on $m, l$ were obtained in [16].

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