ABELIAN THEOREMS FOR A CLASS OF PROBABILITY DISTRIBUTIONS IN $\mathbb{R}^d$
AND THEIR APPLICATION

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A class of multidimensional absolutely continuous distributions is considered. Each of them has a moment-generating function that is finite in a bounded set $S$ and, therefore, generates a family of so-called conjugate or associated distributions. At the focus of our attention are the limiting distributions for this family that appear as the conjugating parameter tends to the boundary of $S$. As in the one-dimensional case, each such limiting distribution can be obtained as a consequence of an Abelian theorem.

1. Introduction

Let $P$ be a probability measure defined on the Borel sets of $\mathbb{R}^d$, $d > 1$, and $f(s)$ be its moment-generating function, that is,

$$f(s) = \int_{\mathbb{R}^d} e^{\langle s, x \rangle} P(dx).$$

By $\langle \cdot, \cdot \rangle$ we denote the inner product. Suppose that the set $S = \{s \in \mathbb{R}^d : f(s) < \infty\}$ is not empty and its dimensionality equals $d$.

The moment-generating function plays a role of great importance in the large-deviation theory. Its basic properties are discussed in [1-5]. The present paper aims to make a contribution toward the further development of this theory. At the focus of our attention is the case when $S$ is, being always convex, bounded.

Let $S_0$ be the interior of $S$. If $S$ is bounded and $0 \in S_0$, then $S_0$ can be represented as

$$S_0 = \{s : s = te, \; 0 \leq t < h(e), \; e \in S^{d-1}\}.$$

It is convenient to call $h(e)$ the shape function of $S$ or simply the shape of $S$.

Obviously, for $0 < t < h(e)$, $u > 0$ the Markov inequality holds, that is,

$$P(x : \langle e, x \rangle \geq u) \leq f(te) e^{-tu}. \quad (1.1)$$

In what follows, we assume that $P$ is absolutely continuous. Denote its density by $p(x)$.

Further, assume that

$$p(x) = b(x) e^{-\|x\| a(e)}, \quad (1.2)$$

where $e_x = |x|^{-1} x$ and

$$0 < \inf_{e \in S^{d-1}} a(e) \leq \sup_{e \in S^{d-1}} a(e) < \infty.$$

If $p(x)$ is of the form (1.2) and $b(x)$ does not grow too fast as $|x| \to \infty$, then $f(s)$ is finite for some $S$ with $0 \in S_0$. Intuitively, it is $a(e)$ that determines the shape of $S$. The following proposition justifies this conjecture.

**Proposition 1.1.** Assume that in (1.2)

$$c_-(1 + |x|)^{-\beta} \leq b(x) \leq c_+(1 + |x|)^{\beta}, \; \beta > 0, \; c_+ > 0. \quad (1.3)$$

Then

1°.

$$h(e) = \inf_{\langle e, \varepsilon \rangle > 0} a(e).$$

2°. For the shape function $h(e)$ of any bounded open convex set $S_0$ that contains 0, there exists $p(x)$ of the form (1.2) such that the interior of $S = \{s : f(s) < \infty\}$ is $S_0$. As $a(e)$ in (1.2) one may take

$$a(e) = \sup_{\langle e, \varepsilon \rangle > 0} h(e)(e, \varepsilon).$$


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The question arises: What can be said about the asymptotic behavior of $f(s)$ as $s \to 0^+$? The answer requires additional restrictions imposed on both $a(e)$ and $b(x)$ in (1.1). Our goal is to establish a multidimensional analog of the following fact.

Let $d = 1$ and

$$p(x) = e^{-s x^2} r_0(x),$$

where $s_+ > 0$ and $r_0(x)$ is of regular variation as $x \to \infty$ with the exponent $\alpha > -1$. When $\tau \downarrow 0$,

$$f(s_+ - \tau) \sim \Gamma(1 + \alpha) \tau^{-1} r_0(\tau^{-1}).$$

This is one of the simplest forms of the so-called Abelian theorem (see, e.g., [12]).

First, we need a relevant multidimensional analog of (1.4) and (1.5). Having it in mind, we introduce the following notion of regular variation that, in essence, coincides with that given in [11, Sec. 5.4.2].

Let $\lambda(e)$ be a nonnegative function defined on $S^{d-1}$.

**Definition 1.2.** We say that $b(x)$, $x \in \mathbb{R}^d$, is the function of $(\alpha, \lambda)$-regular variation in the cone $C_\lambda = \{x \in \mathbb{R}^d : \lambda(e_x) > 0\}$ if

$$b(x) = r_\alpha(|x|)(\lambda(e_x) + u(x)),$$

where $r_\alpha(t)$ is of regular, in Karamata's sense, variation as $t \to \infty$ with the exponent $\alpha$ while

$$\limsup_{|\epsilon| \to \infty} \epsilon \in C_\lambda |u(\epsilon)| = 0.$$

Denote $\Delta(e) = a(e) - h(e)(e, \epsilon)$. We need the following assumptions:

(A) For a given direction $e$, the set $\text{argmin}_{\epsilon \in S^{d-1}} (e, \epsilon) > 0 a(e)/(e, \epsilon)$ consists of a single point $e' = e'(e)$.

(B) $\Delta(e)$ in a neighborhood of $e'$ admits the representation

$$\Delta(e) = \frac{1}{2}(e - e')^T \Lambda(e - e') + o(|e - e'|^2).$$

Here $\Lambda$ is a nonnegative definite matrix, and its rank equals $d - 1$. Furthermore, $\Lambda e' = 0$.

(C) For all sufficiently small $\delta$,

$$\inf_{|\epsilon - e'| > \delta} \Delta(e) = c(\delta) > 0.$$

By $\lambda_j$, $j = 1, \ldots, d - 1$, we denote the nonzero eigenvalues of $\Lambda$.

Consider the class of densities of the form (1.2), where $b(x)$ is of $(\alpha, \lambda)$-regular variation in $(x : |e_x - e| < \delta)$, while in $(x : |e_x - e| > \delta)$ we have

$$b(x) \leq (1 + |x|)^{\beta}$$

for some $\beta > 0$.

**Theorem 1.3.** Let $p(x)$ be of the form (1.2) with $\alpha > -(d + 1)/2$. Assume that $\lambda(e)$ is continuous for $|e - e'| < \delta$ and (A), (B), and (C) hold. Then as $\tau \downarrow 0$ (cf. (1.5)),

$$f(\langle h(e) - \tau \rangle e) = c_\alpha g(e) \tau^{-(d + 1)/2} r_\alpha(\tau^{-1})(1 + o(1)),$$

where

$$c_\alpha = \Gamma\left(\alpha + \frac{d + 1}{2}\right)(2\pi)^{(d-1)/2}$$

and

$$g(e) = \lambda(e')/(e', e')^{-\alpha} - (d + 1)/2(\lambda_1 \cdots \lambda_{d-1})^{-1/2}.$$

The question arises: How does $f(s)$ behave as $s$ approaches $0^+$ alongside some other direction? It turns out that there exists a cone of admissible directions in which the form of the Abelian theorem is, in essence, preserved.

**Theorem 1.4.** If the conditions of Theorem 1.3 hold, then, for any arbitrarily small $\eta > 0$ and $\tau \downarrow 0$,

$$f(h(e)e - \tau \hat{e}) = c_\alpha g(e, \hat{e}) \tau^{-(d + 1)/2} r_\alpha(\tau^{-1})(1 + o(1))$$

uniformly in $\hat{e}, \langle \hat{e}', \hat{e} \rangle \geq \eta$. Here

$$g(e, \hat{e}) = \lambda(e')(\hat{e}', \hat{e})^{-\alpha} - (d + 1)/2(\lambda_1 \cdots \lambda_{d-1})^{-1/2}.$$