THE CLASS $I_0$ ON ABSTRACT STRUCTURES

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We consider the class $I_0$ of elements without indecomposable factor in different abstract structures and prove in these cases that $I_0$ is strictly included in the class of infinitely divisible elements. The structures investigated are convex cones, $q$-probability, and statistical experiments.

1. Introduction

The class $I_0(R)$ is given as the set of those probability measures on $R$ that have no indecomposable factor (in the sense of convolution). It has been shown by Hincin that every measure $\mu \in I_0(R)$ is infinitely divisible (cf. [10, Theorem 5.4.2]). Let us sketch the main ideas of the proof of this theorem. One major ingredient is Hincin's functional

$$N_\alpha(\phi) := - \int_0^1 \log |\phi(u)| \, du$$

for a characteristic function $\phi(u)$ that does not vanish on the interval $[0, \alpha]$. Convolution of distributions corresponds to addition of the Hincin functionals. Furthermore, if $N_\alpha(\varphi_n) \to 0$ ($n \to \infty$), then the concentration functions of $\varphi_n$ tend to 1: $\sup_{x<\alpha} P(x \leq X_n \leq x + \epsilon) \to 1$ ($n \to \infty$) (where $X_n$ denotes a random variable with characteristic function $\varphi_n$). Let $\alpha$ be small enough so that the characteristic function of $\mu$ does not vanish on $[0, \alpha]$. In order to prove [10, Theorem 5.4.2], one first shows (by the compactness of the set of all components of a given fixed distribution) that every component of $\mu$ has a component whose Hincin functional is arbitrarily small. Now, by arranging the Hincin functionals in increasing order, one can show (indirectly by using once again the above-mentioned compactness property) that for every $n \geq 1$ there exists a factorization into $n$ components whose characteristic functions on $[0, \alpha]$ all coincide. By using the theorem on infinite divisibility of limits of infinitesimal triangular systems of probability measures, one arrives at the assertion. The inclusion of $I_0(R)$ in the class of all infinitely divisible laws is strict (cf. [12, (9.4)]): a counterexample of an infinitely divisible probability distribution that is not in $I_0(R)$ is the compound Poisson distribution given by the characteristic function

$$\phi(u) = \exp \left( \log(1 - a) - \log(1 - ae^{iu}) \right) = \exp \left( \sum_{n=1}^{\infty} \frac{a^n}{n} (e^{iun} - 1) \right)$$

for some $0 < a < 1$.

In this paper, we will carry over this inclusion property to random increasing upper semicontinuous functions $g : [0, 1] \to R$, to the framework of $q$-probability, and to statistical experiments. In all these cases, we will show by a counterexample that the inclusion is strict.

2. The Convex Cone of Nonnegative Increasing Upper Semicontinuous Functions

Let $C = USC([0, 1], R_+)$ be the convex cone of nonnegative increasing upper semicontinuous functions on $[0, 1]$ with the topology given in [8, Sec. 6.3]. Infinitely divisible laws on convex cones, in particular on $C = USC([0, 1], R_+)$, were considered in [8, 9]. Our methods will be based on a suitable combination of these works and the classical proof of Theorem 5.4.2 in [10]. As far as the theorem on infinite divisibility of limits of infinitesimal triangular systems $\Delta$ is concerned, the classical proof consists of showing that the row products of $\Delta$ can be approximated by compound Poisson laws, which then automatically gives the infinite divisibility of the limit. Another proof was suggested in [4] (see also [6, (6.6)]), which showed directly that it is possible for any $p \geq 1$ to split up the rows of $\Delta$ into $p$ approximately equal convolution products. We will apply Carnal's method (see [4]) to carry over this theorem to $C$.

Definition 1. A triangular array $\{\mu_{n,j}\}_{n \geq 1, 1 \leq j \leq k(n)}$ of probability measures on $C = USC([0, 1], R_+)$ is called infinitesimal if for any Borel neighborhood $U$ of 0 we have

$$\inf_{1 \leq j \leq k(n)} \mu_{n,j}(U) \to 1, \quad n \to \infty.$$
The convolution product $\ast$ on $C$ is with respect to ordinary addition $+$. Let $M^1(C)$ be the topological monoid of probability measures on $C$ with operation $\ast$ and the weak topology. A $C$-valued probability distribution $\mu$ is called infinitely divisible if for any $p \geq 1$ there is a $C$-valued probability measure $\mu^p$ such that $\mu^p = \mu$. A $C$-valued probability measure $\mu$ is called the limit of a triangular array $(\mu_{n,j})_{n \geq 1; 1 \leq j \leq k(n)} \subset M^1(C)$ if it is the weak limit of row convolution products:

$$\mu_{n,1} \ast \mu_{n,2} \ast \cdots \ast \mu_{n,k(n)} \xrightarrow{w} \mu, \quad n \to \infty.$$ 

A character on $C$ is a continuous function $\chi: C \to [0, 1]$ such that $\chi(x + y) = \chi(x)\chi(y)$ ($x, y \in C$) and $\chi(0) = 1$. Let $X$ be the monoid of characters on $C$. A submonoid $Y$ of $X$ is called plain, if it is separating (i.e., for $x, y \in C$ with $x \neq y$ there is a $\chi \in Y$ such that $\chi(x) \neq \chi(y)$), $1 \in Y$, and for every $x \in C$ there is a $\chi \in Y$ such that $\chi(x) \neq 0$. The Laplace transform of $\mu \in M^1(C)$ is given by $L\mu(\chi) = \int_C \chi(x) \mu(dx)$. The following continuity property holds (cf. [9, Lemma 3.1]):

**Lemma 1.** Let $Y$ be a plain submonoid of $X$. Assume that $\mu$ and $\mu_n$ ($n \geq 1$) are probability measures on $C$. Then $\mu_n \xrightarrow{w} \mu$ if and only if $L\mu_n(\chi) \to L\mu(\chi)$ for every $\chi \in Y$.

Let $\mathcal{F} := \{\lambda(x): C \to \mathbb{R}_+: \lambda(x) = \int_0^1 x \lambda(dx): \lambda \in M^b_{+, \text{cont}}\}$,

where $M^b_{+, \text{cont}}$ denotes the class of bounded nonnegative measures on $[0, 1]$ that are continuous on $[0, 1]$. By [9, (4.3)], $\mathcal{F}$ is a plain submonoid of $X$. In order to apply Carathéodory's method, we must supply a countable plain submonoid of $X$. For $0 \leq a \leq b < 1$, let $U_{a,b}$ be the uniform probability measure on the interval $[a, b]$. Let $L$ be the class of finite convex combinations of measures $U_{a,b}$, $a, b \in Q$, $0 \leq a \leq b \leq 1$. Then the following lemma can be easily verified:

**Lemma 2.** Let

$$\mathcal{F}_0 := \{\lambda(x): C \to \mathbb{R}_+: \lambda(x) = \int_0^1 x \lambda(dx): \lambda \in L\}.$$

Then $\mathcal{F}_0 = e^{-\mathcal{F}}$ is a plain submonoid of $X$.

By looking at the Laplace transform, it follows that for any $p \geq 1$ the set of $p$th convolution roots in $M^1(C)$ of some fixed $\mu \in M^1(C)$ is weakly compact.

**Theorem 1.** Every limit distribution of an infinitesimal triangular array $(\mu_{n,j})_{n \geq 1; 1 \leq j \leq k(n)}$ of probability measures on $C = USC([0, 1], R_+)$ is infinitely divisible on $C$.

**Proof.** Let $p \geq 1$ be fixed and write $F_0 := \{\lambda_n(x)\}_{n \geq 1}$. Let $\varepsilon > 0$ and $Q \geq 1$. As in the proof of [6, Remark 6.5.3 and Theorem 6.6.4], there exists $n_0 = n_0(Q, \varepsilon)$ such that for every $n \geq n_0$ one can decompose the index set $\{1, 2, \ldots, k(n)\}$ into $p$ disjoint classes $\Pi_1^{(n)}, \Pi_2^{(n)}, \ldots, \Pi_p^{(n)}$ with the property

$$\left| \prod_{j \in \Pi_1^{(n)}} L\mu_{n,j}(e^{-\lambda_1(x)}) - \prod_{j \in \Pi_2^{(n)}} L\mu_{n,j}(e^{-\lambda_2(x)}) \right| < \varepsilon, \quad 1 \leq q \leq Q, \quad 1 \leq r, s \leq p.$$ 

Now (as in the proof of [6, Theorem 6.6.4]) setting $\varepsilon_Q := 1/Q$ and $n_Q := n_0(Q, \varepsilon_Q)$ and going over to a suitable subsequence, one gets, by Lemmas 1 and 2, the assertion.

**Definition 2.** The class $I_0(C)$ on $C = USC([0, 1], R_+)$ is defined as the set of those probability measures $\mu \in M^1(C)$ such that every nondegenerate convolution factor of $\mu$ is itself decomposable into a convolution product of two nondegenerate probability measures in $M^1(C)$.

For a probability measure $\mu \in M^1(C)$ for random elements $g \in C$, let $\mu_g \in M^1(R_+)$ be the distribution of $g(x)$ ($x \in [0, 1]$).

**Proposition 1.** Suppose $\mu \in I_0(C)$ such that $\mu_1$ is degenerate. Then $\mu$ is degenerate.

**Proof.** 1. Since the random elements $g \in C$ are nonnegative and increasing, the supports of all the $\mu_x$ must be bounded under our assumptions. We suppose that $\mu$ is nondegenerate and show that $\mu$ and thus every $\mu_x, x \in [0, 1]$, must also be infinitely divisible, which is not possible since the only infinitely divisible laws on $R$ with bounded support are the degenerate ones (this follows from the Lévy–Hinčin formula, cf. [10, (6.1)]).

2. For $\ell \in N$, denote by $\pi_\ell \in M^1(R_+)$ the elementary Poisson measure with parameter $1/\ell$ on $N_0$, and let $\varpi_\ell$ be the probability measure on $C$ given as $(\varpi_\ell)_x = \delta_0 \in M^1(R_+)$ for $0 \leq x < 1$ and $(\varpi_\ell)_x = \pi_\ell \in M^1(R_+)$. Then, under our assumption, obviously $\mu_\ell = \mu \ast \varpi_\ell$ is in $I_0(C)$ and $\mu_\ell \xrightarrow{w} \mu$, $\ell \to \infty$, so $\mu$ remains infinitely divisible if every $\mu_\ell$ is. Thus, it suffices to verify for $\ell \geq 1$ the infinite divisibility of $\mu_\ell$. 

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