SHARP INEQUALITY FOR DEVIATION OF ROGOZINSKI SUMS AND THE SECOND CONTINUITY MODULUS IN THE SPACE OF PERIODIC CONTINUOUS FUNCTIONS

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The sharp constant (uniformly in $n$) is found in a Jackson-type inequality involving the Rogozinski sums of order $n$ and the second modulus of continuity with step $\pi/(n+1)$. Bibliography: 6 titles.

§1. INTRODUCTION

1. Notation. Below, the following notation is adopted: $C$ is the space of $2\pi$-periodic continuous real-valued functions $f$ with the norm $\|f\| = \sup_{t \in \mathbb{R}} |f(t)|$; $\mathcal{H}_n$ is the set of trigonometric polynomials of order not exceeding $n$; $f_\epsilon(t) = f(x + t)$ is the shift of a function $f$ at a number $x$; $f_e(t) = \frac{f(t) + f(-t)}{2}$ is the even part of a function $f$; $P$ is a seminorm on a subspace $\mathcal{M}$ of the space $C$ satisfying the conditions

(a) for any $f \in \mathcal{M}$ and $x \in \mathbb{R}$, the function $f_e \in \mathcal{M}$ and the equality $P(f_e) = P(f)$ holds;
(b) there exists a constant $M$ such that $P(f) \leq M\|f\|$ for $f \in \mathcal{M}$;

$\delta^2_{n}(f, x) = f(x + t) - 2f(x) + f(x - t)$ is the second-order central difference for a function $f$ with step $t$ at a point $x$;

$\omega_2(f, h)_P = \sup_{|t| \leq h} P(\delta^2_{n}(f))$ is the second continuity modulus of a function $f \in \mathcal{M}$ with step $h$ with respect to a seminorm $P$;

$E_n(f) = \inf_{T \in \mathcal{H}_n} \|f - T\|$ is the best approximation of order $n$ for a function $f \in C$.

We omit the index $P$ if $(\mathcal{M}, P) = (C, \| \cdot \|)$. We apply the symbols $\|f\|$ and $\omega_2(f, h)_P$ to discontinuous functions $f$ in the same sense. We also denote

$$\sum_{k=1}^{m} a_k = \frac{a_0}{2} + \sum_{k=1}^{m} a_k;$$

$$\sum_{k=1}^{m} a_k = \frac{a_0}{2} + \sum_{k=1}^{m-1} a_k + \frac{a_m}{2}; \quad \sum_{k=l}^{m} \varphi(x) = \sum_{k=l}^{m} \varphi\left(\frac{k}{n}\right);$$

if a function $\varphi$ is not defined, say, at zero, but has the limit there, then $\varphi(0)$ denotes the value $\lim_{x \to 0} \varphi(x)$;

$$R_n(t) = \frac{1}{\pi} \sum_{k=0}^{n} \cos \frac{k\pi}{2(n+1)} \cos kt = \frac{1}{2\pi} \sin \frac{\pi}{2(n+1)} \cos \left(\frac{n+1/2}{n+1} t\right)$$

$s$ the Rogozinski kernel of order $n$; $\mathcal{R}_n(f, x) = \frac{\pi}{\pi} \int f_x R_n$ is the Rogozinski sum of order $n$ for a function $f$;

$$S_h(t) = \left\{ \begin{array}{ll}
\frac{1}{h} \left(1 - \frac{|t|}{h}\right) & \text{if } |t| \leq h, \\
0 & \text{if } h \leq |t| \leq \pi.
\end{array} \right.$$
is the kernel of the Steklov function of second order with step \( h \); the Bernoulli numbers \( B_n \) are defined by the equality
\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}; \quad B_n(x) = \sum_{k=0}^{n} C_k B_k x^{n-k}
\]
are the Bernoulli polynomials; \( \sin \) is the integral sine; \( \lfloor x \rfloor \) is the integer part of a number \( x \); \( G = \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} \, dx = 0.915965594 \ldots \) is the Catalan constant.

2. Survey of results. The value
\[
D(U, h)_p = \sup_{f \in \mathcal{H}} \frac{P(U(f) - f)}{\omega_2(f, h)_p}
\]
(\( 0 \) is considered to be equal to zero), where \( U : \mathcal{H} \to \mathcal{H}, h > 0 \), is usually called the sharp constant in the inequality
\[
P(U(f) - f) \leq K \omega_2(f, h)_p.
\]
If \( U_n : \mathcal{M} \to H_n, \gamma > 0 \), then equalities of the form
\[
P(U_n(f) - f) \leq K \omega_2\left(f, \frac{\gamma \pi}{n+1}\right)_p
\]
are called Jackson-type inequalities (for the second continuity modulus). In this case, one can pose the problem of finding a constant in inequality (1) that is sharp for all \( n \) simultaneously, i.e., the value
\[
\sup_{n \in \mathbb{Z}^+} D(U_n, \gamma \pi/(n+1))^p.
\]
Earlier the author found \([1, 2]\) the constants \( D(U_n, h) \) for some positive operators \( U_n \). As for the constants that are sharp for all \( n \) simultaneously, only one result of this type was known. In 1974, Zhuk \([3]\) established the inequality
\[
\|f - V_n(f)\| \leq 1 \cdot \omega_2\left(f, \frac{\pi}{2(n+1)}\right)
\]
(where \( V_n \) is a linear operator from \( C \) into \( H_n \)). Later Shalaev \([4]\) discovered that the constant 1 is sharp for all \( n \) simultaneously not only in inequality (2) but also in the inequality
\[
E_n(f) \leq 1 \cdot \omega_2\left(f, \frac{\pi}{4(n+1)}\right).
\]
Namely, it was shown that
\[
\sup_{n \in \mathbb{Z}^+} \sup_{f \in \mathcal{C}} \frac{E_n(f)}{\omega_2\left(f, \frac{\pi}{2(n+1)}\right)} = \sup_{n \in \mathbb{Z}^+} \sup_{f \in \mathcal{C}} \frac{\|f - V_n(f)\|}{\omega_2\left(f, \frac{\pi}{2(n+1)}\right)} = 1.
\]
Thus, the sharp constant 1 in inequality (3) is realized by a sequence \( \{V_n\} \) of linear operators.

In the same paper \([3]\), Zhuk applied the same method (adding and subtracting the second Steklov function) to establish the inequality
\[
\|\mathcal{R}_n(f) - f\| \leq \frac{5}{8} \omega_2\left(f, \frac{\pi}{n+1}\right).
\]
Later he transferred these estimates to the case of an arbitrary space \((\mathcal{M}, P)\) (the results mentioned can be also found in the monograph \([5]\)). Set \( D_n = D(\mathcal{R}_n, \frac{\pi}{n+1}) \). Obviously, \( D_0 = 1/2 \). We show in this paper that the constant 5/8 is not sharp in inequality (4). For the values \( D_n \) with \( n \geq 2 \) we establish estimates of the form \( C'_n \leq D_n \leq C_n \) such that \( \lim_{n \to \infty} C'_n = \lim_{n \to \infty} C_n = \sup_{n \geq 2} C_n = D \). The value \( D \) is evaluated. We also give (without a proof) the established value of \( D_1 \). The constant \( D \) is sharp for all \( n \) simultaneously in inequality (4). Our estimates from above for \( D(\mathcal{R}_n, \frac{\pi}{n+1})_p \) hold for any space \((\mathcal{M}, P)\) with the described properties.