DISCRETE SPECTRUM IN SPECTRAL GAPS OF A SELF-ADJOINT OPERATOR UNDER UNBOUNDED PERTURBATIONS

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Let $A$ be a self-adjoint operator, let $(\alpha, \beta)$ be a gap in the spectrum of $A$, and let $B = A + V$, where, in general, the perturbation operator $V$ is unbounded. We establish some abstract conditions under which the spectrum of $B$ in $(\alpha, \beta)$ is discrete; does not accumulate to $\beta$; is finite. An estimate of the number of eigenvalues of $B$ in $(\alpha, \beta)$ is obtained. Bibliography: 3 titles.

1. Let $A$ be a self-adjoint operator in a Hilbert space $H$. We assume that the spectrum of $A$ contains a (possibly unbounded) gap $(\alpha, \beta)$. We perturb $A$ to obtain an operator $B = B^* = A + V$, where the perturbation $V$ may be unbounded. Several questions of increasing complexity will be discussed. Specifically, we seek conditions sufficient for the spectrum of $B$ (1) to be discrete; (2) not to accumulate to some endpoint of the gap; (3) to be finite. In the latter case, an estimate will be given for the number of eigenvalues (with multiplicity) of $B$ in $(\alpha, \beta)$. The discreteness of the spectrum will be ensured by the classical Weyl theorem on compact perturbations or by its generalizations (see, e.g., [2]). However, for some applications this technique is insufficient. First and foremost, we mean the problems in which $A$ is an elliptic differential operator and $V$ is an operator of higher order. Unfortunately, the lack of space forces us to restrict ourselves to general considerations.

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In what follows, $B(H)$ denotes the space of bounded linear operators on $H$, and $S_\infty(H)$ denotes the space of compact operators on $H$. For a linear densely defined operator $M$, by $D(M)$, $M^*$, $\rho(M)$, and $\sigma(M)$ we denote, respectively, the domain, the adjoint, the resolvent set, and the spectrum of $M$. For a self-adjoint operator $T$ and a Borel subset $\delta$ of the real line, we denote by $E_T(\delta)$ the corresponding spectral projection of $T$, and by $\pi_T(\delta)$ we denote the dimension of the space $E_T(\delta)H$. If $T$ is lower bounded, its greatest lower bound is denoted by $m_T$. We shall use the following definition of subordination, generally adopted in the scattering theory (see [1] and [3]).

A self-adjoint operator $A$ is said to be subordinated to a self-adjoint operator $B$ if there exist continuous functions $f$ and $g$ such that $f(x) \geq 1$ and $g(x) \geq 1$ on $\mathbb{R}$, \(\lim_{x \to -\infty} f(x) = \infty\), and $D(g(B)) \subset D(f(A))$.

2. In this section, we formulate certain results similar to the Weyl theorem concerning the stability of the essential spectrum $\sigma_e(\cdot)$. These results are convenient in the case of perturbations that are not relatively compact. Below, $A$ and $B$ stand for self-adjoint operators on $H$.

**Theorem 1.** If $A$ is subordinated to $B$ and

$$E_B(-a, a)(B - A) E_A(-a, a) \in S_\infty(H)$$

for any $a > 0$, then $\sigma_e(B) \subset \sigma_e(A)$.

**Remark.** Let $a > 0$ and let $z \in \rho(A) \cap \rho(B)$. Then inclusion (1) is equivalent to the relation

$$E_B(-a, a) ((B - zI)^{-1} - (A - zI)^{-1}) E_A(-a, a) \in S_\infty(H).$$

**Theorem 2.** If $A$ is subordinated to $B$, $z \in \rho(A) \cap \rho(B)$, and

$$((B - zI)^{-1} - (A - zI)^{-1}) E_A(-a, a) \in S_\infty(H)$$

for any $a > 0$, then $\sigma_e(B) = \sigma_e(A)$.


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Corollary. If $A$ is subordinated to $B$ and

$$E_A(-a, a) H \subset D(B); \quad (B - A) E_A(-a, a) \in S_\infty(H)$$

for any $a > 0$, then $\sigma_e(B) = \sigma_e(A)$.

Theorem 3. Let $A$, $B$ be lower bounded, let $B > A$ (in the sense of quadratic forms), and let $0 < m_A < \alpha$. If

$$(B^{-1} - A^{-1}) E_A[m_A, \alpha] \in S_\infty(H),$$

then $\sigma_e(B) \cap (\alpha, \beta) = \emptyset$ and $(\beta - \varepsilon, \beta) \subset \rho(B)$ for $\varepsilon > 0$.

3. In what follows, it is assumed that $A$ is a positive-definite operator ($m_A > 0$) and that the perturbation is nonnegative. More precisely, let $a$ be the sesquilinear form of $A$, and let $d[a] = D(A^{1/2})$ be the domain of $a$. Let $D_0 \subset d[a]$ be a linear subspace dense in $H$, and let $W_0$ be a subtended linear operator with $D(W_0) = D_0$. We denote by $W$ the closure of $W_0$ and introduce the form

$$b_0(t)[u, v] = a[u, v] + t(W_0u, W_0v), \quad u, v \in D_0, \quad t > 0.$$  

This form admits a closure $b(t)$, and $d[b(t)]$ does not depend on $t > 0$,

$$\widehat{\Delta} := d[b(t)] \subset d[a] \cap D(W),$$

$$b(t)[u, v] = a[u, v] + t(Wu, Wv), \quad u, v \in \widehat{\Delta}, \quad t > 0.$$  

Let $B(t)$ be the self-adjoint operator on $H$ generated by the form $b(t)$. Now we assume that $0 < m_A < \alpha$ and that

$$E_A[m_A, \alpha] H \subset \widehat{\Delta},$$  

$$WE_A[m_A, \alpha] \in S_\infty(H).$$  

Theorem 4. Conditions (2), (3) imply that the assumptions of Theorem 3 are satisfied by the couple $A, B(t)$; therefore, $\sigma_e(B(t)) \cap (\alpha, \beta) = \emptyset$, and for some $\varepsilon := \varepsilon(t) > 0$ we have $(\beta - \varepsilon, \beta) \subset \rho(B(t))$.

For brevity, we denote $E_1 = E_A[m_A, \alpha], E_2 = E_A[\beta, +\infty), H_k = E_k H, k = 1, 2$. Next, we assume that

$$WE_1 \in B(H)$$  

and introduce the notation

$$A_1 = A|_{H_1}, \quad Q_1 = (WE_1)^*(WE_1)|_{H_1}, \quad \chi(\lambda) = -WE_1(A_1 - \lambda I_1)^{-1}(WE_1)^*, \quad \lambda > \alpha.$$  

Theorem 5. If conditions (2), (3) are fulfilled, then

$$\pi_{B(t)}(\alpha, \beta) \leq \pi_{A_1} + tQ_1(\alpha, +\infty), \quad t > 0.$$  

The operator-valued function (5) is monotone nonincreasing. Assume that it admits a compact majorant,

$$\chi(\lambda) \leq \chi_0 \in S_\infty(H), \quad \lambda > \alpha.$$  

Combining (6) with the Birman–Schwinger principle, we arrive at the following statement.