THE DYNAMICAL INVERSE PROBLEM FOR A NON-SELF-ADJOINT STURM–LIOUVILLE OPERATOR

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An approach to the inverse problem (the so-called BC-method) based on boundary-control theory is developed. A procedure of reconstructing a nonsymmetric matrix-function (a potential) given on a semiaxis by a dynamical response operator is described. The results of numerical tests are presented. Bibliography: 6 titles.

0. INTRODUCTION

In the present paper, an approach to inverse problems (the so-called BC-method) based on boundary control theory [1, 2] is developed. It also gives a new interpretation of the local approach due to A. S. Blagoveshenskii [3]. The BC-method for a non-self-adjoint Sturm–Liouville operator is stated in [5] (see also [6]). In the present paper, a version of this method most suitable for numerical realization is considered. The results of numerical experiments are discussed.

1. THE DIRECT PROBLEM. THE BOUNDARY-CONTROL PROBLEM

1.1. The direct problem

Let \( V(x), x \geq 0 \), be a real \( N \times N \) matrix-function with continuously differentiable elements. Consider the initial boundary-value problem (Problem 1)

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + V(x)u &= 0, \quad (x,t) \in \mathbb{R}_+ \times (0,T), \quad T > 0, \\
u(x,0) &= u_t(x,0) = 0, \\
u(0,t) &= f(t).
\end{align*}
\]

The solution of this problem is a vector-function \( u = u^f(x,t) \) with values in \( \mathbb{R}^n \). Sometimes, when using physical terminology, we call \( V, f, \) and \( u^f \) a potential, a control, and a wave, respectively.

Let a matrix-function \( w(x,t) \) be a solution of the Goursat problem

\[
\begin{cases}
\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + V(x)w = 0, & 0 < x < t < T, \\
w(0,t) = 0, & w(x,x) = \int_0^x V(s)ds.
\end{cases}
\]

It is known that \( w(x,t) \) is twice continuously differentiable in the domain \( \{(x,t) : 0 \leq x \leq t \leq T \} \). The following statement is easily verified.

Proposition 1.1.

(a) If \( f \in C^2([0,T];\mathbb{R}^N) \) and \( f(0) = f'(0) = 0 \), then Problem 1 has a unique classical solution \( u = u^f(x,t) \). In this case, the representation

\[
u^f(x,t) = \begin{cases} f(t-x) + \int_x^t w(x,s)f(t-s)ds, & \text{for } x < t, \\
0, & \text{for } x \geq t,
\end{cases}
\]

is valid.
(b) For $f \in L_2([0, T]; \mathbb{R}^N)$, the function $u^f(x, t)$ defined by (5) satisfies Eq. (1) in the sense of distribution theory.

In the latter case, we regard $u^f$ as a generalized solution of Problem 1 for controls of the class $L_2([0, T]; \mathbb{R}^N)$. By (5), for any fixed moment $t = \xi$ we have

$$\text{supp } u^f(\cdot, \xi) \subset \Omega^\xi, \quad 0 \leq \xi \leq T,$$

where $\Omega^\xi := [0, \xi]$ is an interval of the $OX$ axis and the inclusion

$$u^f(\cdot, T) \in L_2(\Omega^T; \mathbb{R}^N), \quad f \in L_2([0, T]; \mathbb{R}^N),$$

is valid.

Let $T^T_\xi$ be a delay operator:

$$(T^T_\xi f)(t) := \begin{cases} 0, & 0 \leq t < T - \xi, \\ f(t - (T - \xi)), & T - \xi \leq t \leq T, \end{cases}$$

where $\xi$ is a parameter, $\xi \in (0, T)$;

$$T^T_0 f := f, \quad T^T_0 T := 0.$$

The independence of the potential $V(\cdot)$ from time leads to a known property of the solution $u^f$:

$$u^{f-T}(\cdot, T) = u^f(\cdot, \xi).$$

We note another property of the solution $u^f$, the so-called “localization principle,” that is implied by the hyperbolicity of system (1)-(3). For any fixed $\xi \in (0, T/2)$, the values of the solution $u^f(x, t)$ for $(x, t)$, $0 \leq x \leq \xi,$ $x \leq t \leq 2\xi - x$, are uniquely determined by the values $V(x, \leq \xi$, and they are independent of the behavior of $V(x, > \xi$.

1.2. The boundary-control problem

The statement of the boundary-control problem is as follows: given $a \in L_2(\Omega^T; \mathbb{R}^N)$, it is required to find $f \in L_2([0, T]; \mathbb{R}^N)$ such that

$$u^f(\cdot, T) = a.$$  

This setting naturally follows from relations (6), (7).

**Lemma 1.1.** For any $a \in L_2(\Omega^T; \mathbb{R}^N)$, there exists a unique solution of problem (11).

**Proof.** By (5), the above problem is equivalent to the solution of the equation

$$a(x) = f(T - x) + \int_x^T w(x, s) f(T - s) ds, \quad x \in \Omega^T.$$  

The latter is the Volterra equation of the second kind with respect to $f(T - x)$. The solvability of this equation implies the solvability of the boundary control problem.

2. A DYNAMICAL SYSTEM

2.1. The control operator

In this section, we endow Problem 1 with the attributes of a dynamical system, namely, with spaces and operators. The space of controls $\mathcal{F}^T := L_2([0, T]; \mathbb{R}^N)$ is called the outer space of dynamical system (1)-(3). The space $\mathcal{H}^T := L_2(\Omega^T; \mathbb{R}^N)$ is called the inner one; at each instant of time $t = \xi$, the wave $u^f(\cdot, \xi)$ belongs to $\mathcal{H}^T$ (see (6), (7)). The operator $W^T : \mathcal{F}^T \rightarrow \mathcal{H}^T,$

$$W^T f = u^f(\cdot, T),$$

is called the control operator of the system.