THE DYNAMICAL INVERSE PROBLEM FOR A NON-SELF-ADJOINT STURM–LIOUVILLE OPERATOR

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An approach to the inverse problem (the so-called BC-method) based on boundary-control theory is developed. A procedure of reconstructing a nonsymmetric matrix-function (a potential) given on a semiaxis by a dynamical response operator is described. The results of numerical tests are presented. Bibliography: 6 titles.

0. INTRODUCTION

In the present paper, an approach to inverse problems (the so-called BC-method) based on boundary control theory [1, 2] is developed. It also gives a new interpretation of the local approach due to A. S. Blagovesheenskii [3]. The BC-method for a non-self-adjoint Sturm–Liouville operator is stated in [5] (see also [6]). In the present paper, a version of this method most suitable for numerical realization is considered. The results of numerical experiments are discussed.

1. THE DIRECT PROBLEM. THE BOUNDARY-CONTROL PROBLEM

1.1. The direct problem

Let \( V(x), \ x \geq 0, \) be a real \( N \times N \) matrix-function with continuously differentiable elements. Consider the initial boundary-value problem (Problem 1)

\[
\begin{align*}
\partial^2 u / \partial t^2 - \partial^2 u / \partial x^2 + V(x)u &= 0, \quad (x, t) \in \mathbb{R}_+ \times (0, T), \quad T > 0, \\
u(x, 0) &= u_t(x, 0) = 0, \\
u(0, t) &= f(t). 
\end{align*}
\]

The solution of this problem is a vector-function \( u = u^f(x, t) \) with values in \( \mathbb{R}^n \). Sometimes, when using physical terminology, we call \( V, f, \) and \( u^f \) a potential, a control, and a wave, respectively.

Let a matrix-function \( w(x, t) \) be a solution of the Goursat problem

\[
\begin{cases}
\partial^2 w / \partial t^2 - \partial^2 w / \partial x^2 + V(x)w = 0, & \ 0 < x < t < T, \\
w(0, t) = 0, & \\
w(x, x) = -\frac{1}{2} \int_0^x V(s)ds.
\end{cases}
\]

It is known that \( w(x, t) \) is twice continuously differentiable in the domain \( \{(x, t) : 0 \leq x \leq t \leq T\} \). The following statement is easily verified.

Proposition 1.1.

(a) If \( f \in C^2([0, T]; \mathbb{R}^N) \) and \( f(0) = f'(0) = 0 \), then Problem 1 has a unique classical solution \( u = u^f(x, t) \). In this case, the representation

\[
u^f(x, t) = \begin{cases}
f(t - x) + \int_x^t w(x, s)f(t - s)ds, & \text{for } x < t, \\
0, & \text{for } x \geq t,
\end{cases}
\]

is valid.
(b) For \( f \in L_2([0, T]; \mathbb{R}^N) \), the function \( u^f(x, t) \) defined by (5) satisfies Eq. (1) in the sense of distribution theory.

In the latter case, we regard \( u^f \) as a generalized solution of Problem 1 for controls of the class \( L_2([0, T]; \mathbb{R}^N) \). By (5), for any fixed moment \( t = \xi \) we have

\[
\text{supp } u^f(\cdot, \xi) \subset \Omega^\xi, \quad 0 \leq \xi \leq T,
\]

where \( \Omega^\xi := [0, \xi] \) is an interval of the \( OX \) axis and the inclusion

\[
u^f(\cdot, T) \in L_2(\Omega^T; \mathbb{R}^N), \quad f \in L_2([0, T]; \mathbb{R}^N),
\]

is valid.

Let \( T^T_\xi \) be a delay operator:

\[
(T^T_\xi f)(t) := f(T - t) := \begin{cases} 0, & 0 \leq t < T - \xi, \\ f(t - (T - \xi)), & T - \xi \leq t \leq T, \end{cases}
\]

where \( \xi \) is a parameter, \( \xi \in (0, T) \);

\[
T^T_\xi f := f, \quad T^T_0 f := 0.
\]

The independence of the potential \( V(\cdot) \) from time leads to a known property of the solution \( u^f \):

\[
u^{f-\xi}(\cdot, T) = u^f(\cdot, \xi).
\]

We note another property of the solution \( u^f \), the so-called "localization principle," that is implied by the hyperbolicity of system (1)-(3). For any fixed \( \xi \in (0, T/2) \), the values of the solution \( u^f(x, t) \) for \((x, t), 0 \leq x \leq \xi, x \leq t \leq 2\xi - x, \) are uniquely determined by the values \( V |_{x \leq \xi} \), and they are independent of the behavior of \( V |_{x > \xi} \).

### 1.2. The boundary-control problem

The statement of the boundary-control problem is as follows: given \( a \in L_2(\Omega^T; \mathbb{R}^N) \), it is required to find \( f \in L_2([0, T]; \mathbb{R}^N) \) such that

\[
u^f(\cdot, T) = a.
\]

This setting naturally follows from relations (6), (7).

**Lemma 1.1.** For any \( a \in L_2(\Omega^T; \mathbb{R}^N) \), there exists a unique solution of problem (11).

**Proof.** By (5), the above problem is equivalent to the solution of the equation

\[
a(x) = f(T - x) + \int_x^T w(x, s)f(T - s)ds, \quad x \in \Omega^T.
\]

The latter is the Volterra equation of the second kind with respect to \( f(T - x) \). The solvability of this equation implies the solvability of the boundary control problem.

### 2. A DYNAMICAL SYSTEM

#### 2.1. The control operator

In this section, we endow Problem 1 with the attributes of a dynamical system, namely, with spaces and operators. The space of controls \( \mathcal{F}^T := L_2([0, T]; \mathbb{R}^N) \) is called the outer space of dynamical system (1)-(3). The space \( \mathcal{H}^T := L_2(\Omega^T; \mathbb{R}^N) \) is called the inner one; at each instant of time \( t = \xi \), the wave \( u^f(\cdot, \xi) \) belongs to \( \mathcal{H}^T \) (see (6), (7)). The operator \( W^T : \mathcal{F}^T \rightarrow \mathcal{H}^T \),

\[
W^T f = u^f(\cdot, T),
\]

is called the control operator of the system.