SOLVING THE VENTTSSEL PROBLEM FOR THE LAPLACE AND
HELMHOLTZ EQUATIONS WITH THE HELP OF ITERATED
POTENTIALS

V. V. Lukyanov and A. I. Nazarov

Using the properties of iterated potentials, the Venttsel problem for the Laplace and Helmholtz equations is
reduced to the Fredholm equation. The possibility of applying the iterated potentials in studies of the Helmholtz
equation with boundary conditions of higher orders is shown. Bibliography: 7 titles.

The Venttsel boundary condition for second-order elliptic equations was stated originally in [1] in studying
diffusion problems. This condition specifies an elliptic operator on the boundary of a domain via the
tangential variables, in combination with the normal derivative. Consider the simplest mechanical problem
in which this condition arises.

Let an elastic isotropic membrane execute small vertical oscillations. In addition, a cord (a string) is
passed through the edge of it, which, in turn, can oscillate in the vertical direction. Using the Hamilton–
Ostrogradskii principle, it is easy to conclude that the standard equation of oscillations

$$p(x)u_{tt} - \text{div}(T(x)\nabla u) = f(x, t)$$

will be accompanied by the following boundary condition:

$$p_1(x)u_{tt} - \frac{\partial}{\partial \tau} \left( T_1(x)\frac{\partial u}{\partial \tau} \right) + T(x)\frac{\partial u}{\partial n} = f_1(x, t).$$

Here, $p$ and $p_1$ are the surface density of the membrane and the linear density of the string, respectively;
$T$ and $T_1$ are their tension coefficients; $f$ and $f_1$ are the densities of distribution of outer forces applied to
them; $\tau$ and $n$ are, respectively, the tangent and normal directions on the boundary of the domain.

In the case where the membrane and string are homogeneous, the consideration of static problem (0.1)–
(0.2) leads to the Venttsel problem for the Laplace equation, but the consideration of the stationary problem
leads to a respective problem for the Helmholtz equation.

At present, many papers are devoted to the study of the Venttsel problem for linear and nonlinear
equations (for a historic review (not complete), see [2, 3]). Usually, the general methods of solving elliptic
problems in the Hölder and Sobolev spaces are used. The desire to obtain a solution of the problem for
continuous boundary conditions leads to the idea of using integral equations of potential theory.

The paper is subdivided into four parts. In Sec. 1, the statement of the inner and outer Venttsel problems
is given, and uniqueness theorems are proved for them, as well as some necessary solvability conditions. In
Sec. 2, properties of iterated potentials of simple and double layers are studied. In Sec. 3, the problems
$\mathcal{W}_i$ and $\mathcal{W}_e$ are reduced to integral equations whose solvability is proved. Section 4 is devoted to the
application of the iterated potentials to the solution of other boundary-value problems with boundary
conditions of higher order.

Notation

By $S$ we denote a closed smooth surface in $\mathbb{R}^m$ (in Secs. 2 and 3 we assume that $S \in C^2$); $n$ is the unit
vector of the outer normal to the surface $S$; $\Omega_i$ is the domain bounded by the surface $S$; $\Omega_e = \mathbb{R}^m \setminus \Omega_i$;
$S_\delta$ is a surface parallel to $S$; $\Omega_\delta$ is the domain bounded by the surface $S_\delta$; $\chi_m = \frac{2m^{m/2}}{\Gamma(\frac{m}{2})}$ is the area of the unit
sphere in $\mathbb{R}^m$; $B_R(x)$ is a sphere of radius $R$ with center at the point $x$; $B_R = B_R(0)$.

November 27, 1997.
By $G_k(x)$ we denote a fundamental solution of the operator $-(\Delta + k^2)$, $k \geq 0$:

$$-(\Delta + k^2)G_k = \delta(x), \quad G_k = G_k(|x|),$$

satisfying some radiation condition at infinity. We use the following notation for the potentials of simple and double layers:

$$V^\mu_k(x) = \int_S \mu(y)G_k(x - y)dS_y, \quad W^\mu_k(x) = \int_S \mu(y)\frac{\partial}{\partial n_y}G_k(x - y)dS_y.$$

The functions $V^\mu_{k_1,k_2}(x) = V^\mu_{k_2}(x)$, $W^\mu_{k_1,k_2}(x) = W^\mu_{k_2}(x)$ are called the iterated potentials of a simple and a double layer, respectively. In the case where the values of $k_1$ and $k_2$ are not important to us, we denote the potentials by $V^\mu_\omega$ and $W^\mu_\omega$.

By $\Delta$ we denote the Beltrami operator on the surface $S$:

$$\Delta u = \Delta u|_S - \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial n} \left( \sqrt{\gamma} \frac{\partial u}{\partial n} \right),$$

where $\gamma$ is the determinant of the metric tensor of the surface $S$. Since in the sequel we consider functions that are insufficiently smooth in $\Omega$ ($\Omega = \Omega_i$ or $\Omega = \Omega_e$), we now introduce the notion of a regular Beltrami operator on $S$ (similar to the notion of a regular normal derivative). Let $u \in C^2(\Omega_i) \cap C^1(\Omega_i)$. If a uniform limit

$$\lim_{\delta \to 0} \Delta u(x - \delta n_x), \quad x \in S,$

exists (here the Beltrami operator on the surface $S_\delta$ is under the sign of limit), then this limit is called the value of the inner regular Beltrami operator of the function $u$ on $S$ and is denoted by $\Delta_i u$. The outer regular Beltrami operator is defined similarly and is denoted by $\Delta_e u$.

## 1. STATEMENT OF THE PROBLEM

It is required to find a function $u \in C^2(\Omega) \cap C^1(\Omega)$ ($\Omega = \Omega_i$ or $\Omega = \Omega_e$) satisfying the Helmholtz equation (or satisfying the Laplace equation for $k = 0$)

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega,$$

having a regular Beltrami operator on $S$ and satisfying the following boundary condition: for $\Omega = \Omega_i$,

$$\Delta_i u - \alpha \left( \frac{\partial u}{\partial n} \right)_i + \beta k^2 u = f \quad \text{on} \quad S, \quad \alpha, \beta \in \mathbb{R}_+, \quad f \in C(S);$$

for $\Omega = \Omega_e$,

$$\Delta_e u + \alpha \left( \frac{\partial u}{\partial n} \right)_i + \beta k^2 u = f \quad \text{on} \quad S, \quad \alpha, \beta \in \mathbb{R}_+, \quad f \in C(S).$$

For $\Omega = \Omega_e$, the following additional conditions are imposed at infinity (the radiation conditions): as $|x| \to \infty$,

$$u(x) = O\left( |x|^{-m-1} \right), \quad \frac{\partial u(x)}{\partial |x|} + iku(x) = o\left( |x|^{-m} \right), \quad k > 0;$$

$$u(x) = O\left( |x|^{-2-m} \right), \quad k = 0.$$

The problem for $\Omega = \Omega_i$ ($\Omega = \Omega_e$) is called the inner (respectively, outer) Venttsel problem and is denoted by $\langle W_i \rangle$ (respectively, $\langle W_e \rangle$).