Given a Riemannian manifold $M$, a Banach space $E$, and a continuous positive function $\rho : M \rightarrow \mathbb{R}$ (a weight function), define the Banach space $L^k_p(M, \rho; E)$ that consists of measurable $E$-valued differential forms on $M$ of degree $k$ such that $\|\omega\| = (\int_M |\omega(x)|^p \rho(x) d\mu) 1/p < \infty$. The exterior derivative $d^k : L^k_p(M, \rho; E) \rightarrow L^{k+1}_p(M, \rho; E)$ is densely-defined and closed; moreover, $\text{im} d^k \subset \ker d^{k+1}$. The quotient space $\text{im} d^k / \ker d^{k+1}$ is called the $L^p$-cohomology space of $M$ and denoted by $H^k_p(M, \rho; E)$.

Suppose that $X$ and $Y$ are Riemannian manifolds of dimensions $m$ and $n$ and $f$ is a smooth positive function on $X$. The symbol $X \times_f Y$ denotes the warped product of $X$ and $Y$ endowed with the Riemannian metric $d(x, y)^2 = dz^2 + f^2(x) dy^2$. Suppose that $\rho(x)$ and $\sigma(y)$ are weight functions on $X$ and $Y$ respectively. It was proven in [1] that if $p \in ]1, \infty[$, the function $f$ is bounded, and the de Rham $L^p$-complex of $L_p^p(Y, \sigma; \mathbb{R})$ is splitting (i.e., $\text{im} d^j$ and $\ker d^{j+1}$ are complemented subspaces in $L_p^p(Y, \sigma; \mathbb{R})$ for $j \in \mathbb{N}$) then the topological vector isomorphism

$$H^k(X \times_f Y, \rho \sigma; \mathbb{R}) \cong \bigoplus_{i+j=k} H^i_p(X, \rho f^{n/p-j}; H^j_p(Y, \sigma; \mathbb{R}))$$

holds (Künneth's formula).

In the same article, it was established that the de Rham $L^p$-complex on a smooth compact manifold is splitting if $1 < p < \infty$. We do not know whether this complex is splitting for $p \in \{1, \infty\}$. Moreover, even if the de Rham $L^p$-complex on a manifold $Y$ is splitting, we cannot transfer the method for proving Künneth's formula of [1] to the case of $p \in \{1, \infty\}$, since the corresponding $L^p$-spaces of differential forms are not reflexive.

If $p = 2$ then a somewhat different version of Künneth's formula was established by Zucker [2] under some additional conditions on the de Rham complex of $L_2^p(X \times Y)$.

The method developed in the present article enables us to establish Künneth's formula for every $p \in [1, \infty]$ in the case when $Y$ is compact or its de Rham $L^p$-complex is splitting. Moreover, we have managed to remove the smoothness condition on the manifolds by replacing it with the assumption that $X$ and $Y$ are Lipschitz Riemannian manifolds.

1. Differential Forms on a Lipschitz Riemannian Manifold

Suppose that $U$ is an open subset in $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_n \geq 0\}$ and $V$ is a vector space. Given $k \in \mathbb{Z}$, the symbol $\Omega^k(U; V)$ denotes the space of $V$-valued differential forms of degree $k$ on $U$. For $k \geq 0$, each form $\omega \in \Omega^k(U; V)$ has the coordinate representation $\omega = \sum_{|I|=k} \omega^I dx_I$. Here $I = (i_1, \ldots, i_k)$ is a multi-index of length $|I| = k$, $\omega^I \in \Omega^0(U; V)$, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{R}^n$. Moreover, we suppose that if $|I| = 0$, i.e., $I$ is the empty collection of indices, then $dx_I = 1 \in \Omega^0(M; \mathbb{R})$.

We say that differential forms are equal if their coefficients coincide almost everywhere.

Suppose that $T : V \rightarrow V'$ is a linear mapping of vector spaces. Given a form $\omega = \sum_{|I|=k} \omega^I dx_I$, we put $T \omega = \sum_{|I|=k} (T \omega^I) dx_I$.

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Henceforth the letter $E$ stands for a Banach space. A form $\omega \in \Omega^k(U; E)$ is called measurable if its coefficients are measurable functions (by a measurable function we mean a function that is almost everywhere the limit of some sequence of step functions). The symbol $L^k(U; E)$ denotes the set of forms in $\Omega^k(U; E)$ with locally integrable coefficients.

We say that a form $\theta \in L^{k+1}(U; E)$ is the differential of a form $\omega \in L^k(U; E)$ and write $\theta = d\omega$ if the equality
\[
\int_U \alpha \wedge \theta = (-1)^{\alpha+1} \int_U d\alpha \wedge \omega
\]
holds for every smooth real form $\alpha \in L^{n-k-1}(U; \mathbb{R})$ whose support is compact and lies in $\text{int} \ U$. Here and in the sequel, the superscript $\alpha$ denotes the degree of $\alpha$. The symbol $W^k(U; E)$ denotes the set of the forms $\omega \in L^k(U; E)$ for which $d\omega \in L^{k+1}(U; E)$.

Using the standard "smoothing" operation, we can demonstrate that the formula
\[
\int_U d\omega = 0 \tag{1.1}
\]
is valid for every form $\omega \in W^{n-1}(U; E)$ with compact support. Suppose that $U_1$ and $U_2$ are open subsets of $\mathbb{R}^n_+$ and $\varphi : U_1 \to U_2$ is a locally Lipschitz isomorphism. By the Stepanov-Rademacher theorem [3], the mapping $\varphi$ is almost everywhere differentiable. Consequently, the pull-back operation $\varphi^* : \Omega^k(U_2; V) \to \Omega^k(U_1; V)$ is well defined.

In the article [4], the following relations were established:
\[
\varphi^*(L^k(U_2; E)) = L^k(U_1; E); \quad \varphi^*(W^k(U_2; E)) = W^k(U_1; E); \tag{1.2}
\]
\[
d\varphi^* \omega = \varphi^* d\omega \quad \forall \omega \in W^k(U_2; E). \tag{1.3}
\]

Let $M$ be an $n$-dimensional manifold with countable base. A chart on $M$ is a homeomorphism $\varphi : M_\varphi \to U_\varphi$, where $M_\varphi$ is an open subset in $M$ and $U_\varphi$ is an open subset in $\mathbb{R}^n_+$. An atlas $A$ on $M$ is said to be a Lipschitz atlas if the mapping $\varphi\psi^{-1} : (M_\psi \cap M_\varphi) \to (M_\psi \cap M_\varphi)$ is a locally Lipschitz isomorphism for each pair $\varphi, \psi \in A$ of charts.

A manifold $M$ with a maximal Lipschitz atlas is called a Lipschitz manifold and the atlas is denoted by $A_M$.

Suppose that $M$ is a Lipschitz manifold and $B \subset A_M$. We say that a collection $\omega = (\omega_\varphi \in \Omega^k(U_\varphi; V) \mid \varphi \in B)$ of forms is compatible if the equality $(\varphi_\psi^{-1})^* \omega_\varphi(x) = \omega_\psi(x)$ is valid for all $\varphi, \psi \in B$ and almost every $x \in (M_\psi \cap M_\varphi)$.

A compatible collection $\omega = (\omega_\varphi \in \Omega^k(U_\varphi; V) \mid \varphi \in A_M)$ is called a $V$-valued differential form of degree $k$ on $M$; in this case, we write $\omega \in \Omega^k(M; V)$.

Remark. Suppose that $A \subset A_M$ is an atlas on a Lipschitz manifold $M$. Given a compatible collection $(\omega_\varphi \in \Omega^k(U_\varphi; V) \mid \varphi \in A)$, there is a unique form $\tilde{\omega} \in \Omega^k(M; V)$ such that $\omega_\varphi = \varphi_\psi^{-1} \tilde{\omega}_\psi$ for all $\varphi \in A$. Moreover, if each form $\omega_\varphi$ belongs to $L^k(U_\varphi; V)$ $(W^k(U_\varphi; V))$ then the form $\tilde{\omega}_\varphi$ belongs to $L^k(U_\varphi; V)$ $(W^k(U_\varphi; V))$ for each chart $\psi$ in $A_M$. This follows from (1.2).

Suppose that $T : V \to V'$ is a linear mapping. Given a form $\omega \in \Omega^k(M; V)$, put $T\omega = (T\omega)_\varphi (\mid \varphi \in A_M)$. Thereby the mapping $T : V \to V'$ determines the mapping $T : \Omega^k(M; V) \to \Omega^k(M; V')$.

A form $\omega \in \Omega^k(M; E)$ is said to be measurable if the forms $\omega_\varphi \in \Omega^k(U_\varphi; E)$ are measurable for all $\varphi \in A_M$. Also, put
\[
L^k(M; E) = \{ \omega \in \Omega^k(M; E) \mid \omega_\varphi \in L^k(U_\varphi; E), \varphi \in A_M \},
\]
\[
W^k(M; E) = \{ \omega \in \Omega^k(M; E) \mid \omega_\varphi \in W^k(U_\varphi; E), \varphi \in A_M \}.
\]