Asymmetric Decompositions of Abelian Groups

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ABSTRACT. A subset \( A \) of an Abelian group \( G \) is said to be asymmetric if \( g + S \not\subseteq A \) for any element \( g \in G \) and any infinite symmetric subset \( S \subseteq G \) \( (S = -S) \). The minimal cardinality of a decomposition of the group \( G \) into asymmetric sets is denoted by \( \nu(G) \). For any Abelian group \( G \), the cardinal number \( \nu(G) \) is expressed via the following cardinal invariants: the free rank, the 2-rank, and the cardinality of the group. In particular, \( \nu(\mathbb{Z}^n) = n + 1 \), \( \nu(\mathbb{Q}^n) = n + 2 \), and \( \nu(\mathbb{R}) = \aleph_0 \).

KEY WORDS: Abelian group, asymmetric set, asymmetric decomposition, free rank, 2-rank.

A subset \( A \) of an Abelian group \( G \) is said to be asymmetric if \( g + S \not\subseteq A \) for any element \( g \in G \) and any infinite symmetric subset \( S \subseteq G \), \( S = -S \) (in other words, \( A \) contains no infinite subset symmetric with respect to some point).

For any Abelian group \( G \), define \( \nu(G) \) as the minimal cardinality of a decomposition of the group \( G \) into asymmetric sets. In the chromatic terminology, \( \nu(G) \) is the minimal number of colors that are needed for an asymmetric coloring of the group \( G \). It is clear that \( \nu(G) = 1 \) if and only if the group \( G \) is finite.

The number \( \nu(G) \) was introduced by Protasov in [1], where the groups \( G \) with \( \nu(G) = 2 \) were also characterized and the problem of calculating the cardinal \( \nu(G) \) for the specific groups \( \mathbb{Z}^n \), \( \mathbb{Q}^n \), and \( \mathbb{R} \) was posed.

In this note we calculate the invariant \( \nu(G) \) for all Abelian groups \( G \) via the following known cardinal invariants: the free rank \( r_0(G) \), the 2-rank \( r_2(G) \), and the cardinality \( |G| \) of the group \( G \) (for the definitions of the ranks \( r_0(G) \) and \( r_p(G) \) for all primes \( p \), see [2, p. 103 of the Russian translation]). Assuming that all groups under consideration are Abelian, we first present the main results of the paper.

**Theorem 1.** If \( G \) is a finitely generated group, then \( \nu(G) = r_0(G) + 1 \).

**Theorem 2.** If \( G \) is a countable infinitely generated group and the numbers \( r_0(G) \) and \( r_2(G) \) are finite, then \( \nu(G) = r_0(G) + 2 \).

**Theorem 3.** If \( G \) is a countable group and at least one of the numbers \( r_0(G) \) and \( r_2(G) \) is infinite, then \( \nu(G) = \aleph_0 \).

**Theorem 4.** If \( G \) is an uncountable group, then \( \nu(G) = \max\{r_2(G), \log |G|\} \), where we use the notation \( \log |G| = \min\{\gamma : 2^\gamma \geq |G|\} \).

It follows from Theorems 1 and 2 that \( \nu(\mathbb{Z}^n) = n + 1 \) and \( \nu(\mathbb{Q}^n) = n + 2 \) for any positive integer \( n \in \mathbb{N} \). By Theorem 4, \( \nu(\mathbb{R}) = \aleph_0 \).

To prove Theorems 1 and 2, we use some nontrivial results from algebraic topology. Theorem 3 is a simple consequence of Theorem 1, and Theorem 4 is proved by techniques from the combinatorial theory of sets. The structure of the present note is such that the proofs of the main results are taken from a sequence of 16 lemmas.

Let us first somewhat modify the definition of the invariant \( \nu(G) \) and introduce a new invariant \( \tilde{\nu}(G) \). Note that a subset \( A \) of a group \( G \) is asymmetric if and only if the intersection \( A \cap (2g - A) \) is finite for any \( g \in G \). A subset \( A \subseteq G \) is said to be absolutely asymmetric if the intersection \( A \cap (g - A) \) is finite for any \( g \in G \). Let us define the cardinal \( \tilde{\nu}(G) \) as the minimal cardinality of a decomposition of the group \( G \) into absolutely asymmetric subsets. Since any absolutely asymmetric set is asymmetric, it follows that \( \nu(G) \leq \tilde{\nu}(G) \leq |G| \).
The next assertion immediately follows from the definitions of the invariants \( \nu \) and \( \tilde{\nu} \).

**Lemma 1.** If \( H \) is a subgroup of a group \( G \), then \( \nu(H) \leq \nu(G) \) and \( \tilde{\nu}(H) \leq \tilde{\nu}(G) \).

**Lemma 2.** If \( H \) is a subgroup of a group \( G \), then \( \tilde{\nu}(G) \leq \tilde{\nu}(H) \times \tilde{\nu}(G/H) \).

**Proof.** Let us choose decompositions \( H = \bigcup_{\alpha \in \tilde{\nu}(H)} H_\alpha \) and \( G/H = \bigcup_{\beta \in \tilde{\nu}(G/H)} F_\beta \) of the groups \( H \) and \( G/H \) into absolutely asymmetric subsets. Let \( s : G/H \to G \) be an arbitrary section of the quotient mapping \( \pi : G \to G/H \) (i.e., \( \pi \circ s(g) = g \) for any \( g \in G/H \)).

For any pair \( (\alpha, \beta) \in \tilde{\nu}(H) \times \tilde{\nu}(G/H) \), we set \( G_{\alpha,\beta} = H_\alpha + s(F_\beta) \). We can readily see that

\[
G = \bigcup_{(\alpha,\beta) \in \tilde{\nu}(H) \times \tilde{\nu}(G/H)} G_{\alpha,\beta}.
\]

Let us show that any subset \( G_{\alpha,\beta} \) is absolutely asymmetric. To this end, we choose \( g \in G \). Since the subset \( F_\beta \subset G/H \) is absolutely asymmetric, it follows that the intersection \( F_\beta \cap (\pi(g) - F_\beta) \) is finite. Note that \( g - s(\pi(g) - f) - s(f) \in H \) for any \( f \in G/H \). Then, since the set \( H_\alpha \subset H \) is absolutely asymmetric, it follows that the subset

\[
B = \bigcup_{f \in F_\beta \cap (\pi(g) - F_\beta)} H_\alpha \cap (g - s(\pi(g) - f) - s(f) - H_\alpha)
\]

is finite.

We claim that

\[
G_{\alpha,\beta} \cap (g - G_{\alpha,\beta}) \subset B + s(F_\beta \cap (\pi(g) - F_\beta)).
\]

Indeed, for any \( a \in G_{\alpha,\beta} \cap (g - G_{\alpha,\beta}) \), there is an element \( b \in G_{\alpha,\beta} \) such that \( a = g - b \). By the definition of the subset \( G_{\alpha,\beta} \), we have \( a = h_\alpha + s(f_a) \) and \( b = h_\beta + s(f_b) \) for some \( h_\alpha, h_\beta \in H_\alpha \) and \( f_a, f_b \in F_\beta \). Hence, \( h_\alpha + s(f_a) = g - h_\beta - s(f_b) \). Applying the quotient mapping \( \pi \) to this relation, we obtain \( a = \pi(g) - f_b \), i.e., \( f_a \in F_\beta \cap (\pi(g) - F_\beta) \). Let us show that \( h_\alpha \in B \), which will prove (2). Indeed, \( a = h_\alpha + s(f_a) = g - b = g - h_\beta - s(f_b) \). Since \( f_b = \pi(g) - f_a \), it follows that \( h_\alpha = g - s(\pi(g) - f_a) - s(f_a) - h_\beta \), and hence \( h_\alpha \in B \).

It follows from (2) that the set \( G_{\alpha,\beta} \cap (g - G_{\alpha,\beta}) \) is finite. Thus, \( \tilde{\nu}(G) \leq \tilde{\nu}(H) \times \tilde{\nu}(G/H) \). \( \square \)

**Lemma 3.** If \( H \) is a subgroup of a countable group \( G \), then \( \tilde{\nu}(G) \leq \tilde{\nu}(H) + \tilde{\nu}(G/H) - 1 \).

**Proof.** Without loss of generality, we may assume that the numbers \( \tilde{\nu}(H) \) and \( \tilde{\nu}(G/H) \) are finite. Moreover, we may assume that the quotient group \( G/H \) is infinite (otherwise \( \tilde{\nu}(G/H) = 1 \), and we have \( \tilde{\nu}(G) \leq \tilde{\nu}(H) = \tilde{\nu}(H) + \tilde{\nu}(G/H) - 1 \) by Lemma 2).

Let \( G = \{g_n : n \in \mathbb{N}\} \) and \( \{n(k)\}_{k \in \mathbb{N}} \) be a number sequence such that \( G = \bigcup_{k=1}^{\infty} g_{n(k)} + H \) and \( g_{n(i)} + H \neq g_{n(j)} + H \) for \( i \neq j \). Set \( p = \tilde{\nu}(H) \) and \( q = \tilde{\nu}(G/H) \) and choose decompositions \( H = \bigcup_{\alpha=1}^{p} H_\alpha \) and \( G/H = \bigcup_{\beta=1}^{q} F_\beta \) of the groups \( H \) and \( G/H \) into absolutely asymmetric subsets.

For any \( \alpha \in \{1, \ldots, p-1\} \) and \( n \in \mathbb{N} \) we set

\[
A_\alpha(n) = \bigcup_{i,j,k \leq n} (g_{n(i)} + H_\alpha) \cap (g_k - g_{n(j)} - H_\alpha).
\]

Since \( H_\alpha \) is absolutely asymmetric, it follows that the set \( A_\alpha(n) \) is finite.

Now let us construct a decomposition \( G = \bigcup_{\gamma=1}^{p+q-1} G_\gamma \) of the group \( G \) into absolutely asymmetric subsets. For any \( \gamma \in \{1, \ldots, p-1\} \) we set

\[
G_\gamma = \bigcup_{k=1}^{\infty} (g_{n(k)} + H_\gamma) \setminus A_\gamma(k).
\]