First Mixed Problem for a Parabolic Difference-Differential Equation

A. L. Skubachevskii and R. V. Shamin

ABSTRACT. The first mixed boundary value problem for a parabolic difference-differential equation with shifts with respect to the spatial variables is considered. The unique solvability of this problem and the smoothness of generalized solutions in some cylindrical subdomains are established. It is shown that the smoothness of generalized solutions can be violated on the interfaces of neighboring subdomains.

KEY WORDS: parabolic difference-differential equation, strongly elliptic operator, Sobolev spaces.

Introduction

Parabolic functional-differential equations arise in the investigation of nonlinear optic systems with two-dimensional feedback [1-3]. In contrast to parabolic differential equations, these equations have a number of principally new properties. For instance, the smoothness of a generalized solution can be violated inside the cylindrical domain even for an infinitely smooth right-hand side of the equation.

In the present paper, the first mixed problem for a parabolic difference-differential equation with shifts with respect to the spatial variables is considered. The unique solvability of this problem and the smoothness of generalized solutions (in the sense of distributions) in some cylindrical subdomains are established. It is also shown that the smoothness of generalized solutions can be violated on the interfaces of neighboring subdomains.

§1. Statement of the problem

Let \( Q \subset \mathbb{R}^n \) be a bounded domain with boundary \( \partial Q = \bigcup_i M_i \) (\( i = 1, \ldots, N \)), where \( M_i \) are \((n-1)\)-dimensional manifolds of class \( C^\infty \) that are open and connected in the topology of \( \partial Q \). Assume that, in a neighborhood of any point \( g \in \partial Q \setminus \bigcup_i M_i \), the domain \( Q \) is diffeomorphic to an \( n \)-dimensional dihedral angle for \( n > 3 \) and to a plane angle for \( n = 2 \).

Denote by \( W^k_{2}(Q) \) the Sobolev space of complex-valued functions in \( L^2(Q) \) that have generalized derivatives (in the sense of distributions) belonging to \( L^2(Q) \) up to the order \( k \); this space is endowed with the norm

\[
\|u\|_{W^k_{2}(Q)} = \left\{ \sum_{|\alpha| \leq k} \int_Q |D^\alpha u(x)|^2 \, dx \right\}^{1/2}
\]

Denote by \( \hat{W}^k_{2}(Q) \) the closure in \( W^k_{2}(Q) \) of the set \( C^\infty(Q) \) of compactly supported infinitely differentiable functions and by \( W^{-1}_{2}(Q) \) the space dual to \( \hat{W}^1_{2}(Q) \).

Introduce the bounded difference-differential operator \( A_{R}: \hat{W}^1_{2}(Q) \rightarrow W^{-1}_{2}(Q) \) by the formula

\[
A_{R}u = \sum_{i,j=1}^{n} (R_{ij}Q u_{x_j})_{x_i} + \sum_{i=1}^{n} R_{iQ}u_{x_i} + R_{0Q} u.
\]

Here \( R_{ijQ} = P_{Q}R_{ij}I_{Q} \), \( R_{iQ} = P_{Q}R_{iQ}I_{Q} \),

\[
R_{ij}u(x) = \sum_{h \in M} a_{ijh}(x)u(x+h) \quad (i, j = 1, \ldots, n),
\]

\[
R_{i}u(x) = \sum_{h \in M} a_{ih}(x)u(x+h) \quad (i = 0, 1, \ldots, n),
\]

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$M \subset \mathbb{R}^n$ is a finite set of vectors with integral-valued coefficients, $a_{ijk}$, $a_{ih} \in C^\infty(Q)$, $I_Q : L^2(Q) \to L^2(\mathbb{R}^n)$ is the operator of extending any function in $L^2(Q)$ by zero to $\mathbb{R}^n \setminus Q$, and $P_Q : L^2(\mathbb{R}^n) \to L^2(Q)$ is the operator of restricting a function in $L^2(\mathbb{R}^n)$ to $Q$.

**Definition 1.** An operator $-A_R$ is said to be **strongly elliptic** if there are constants $c_1 > 0$ and $c_2 \geq 0$ such that

$$-\text{Re}(A_Ru, u)_{L^2(Q)} \geq c_1\|u\|_{W^2_2(Q)}^2 - c_2\|u\|_{L^2(Q)}^2 \quad (1.2)$$

for any $u \in C^\infty(Q)$.

Necessary and sufficient conditions for strong ellipticity in algebraic form will be stated at the end of this section.

Let us consider a difference-differential equation

$$u_t(z, t) - A_Ru(z, t) = f(z, t) \quad ((z, t) \in \Omega_T) \quad (1.3)$$

with boundary condition

$$u|_{\Gamma_T} = 0 \quad ((z, t) \in \Gamma_T) \quad (1.4)$$

and initial condition

$$u|_{t=0} = \varphi(z) \quad (z \in Q), \quad (1.5)$$

where $\Omega_T = Q \times (0, T)$ and $\Gamma_T = \partial Q \times (0, T)$, $0 < T < \infty$.

Everywhere below we assume that the operator $-A_R$ is strongly elliptic. In this case it is natural to refer to problem (1.3)–(1.5) as the **first mixed problem** for a parabolic difference-differential equation. Without loss of generality we assume that $c_2 = 0$ in inequality (1.2). Indeed, the standard change of the unknown function, $u = \exp(c_2t)w$, reduces Eq. (1.3) to the form $(-A_R + c_2I)w + w_t = \exp(-c_2t)f(z, t) ((z, t) \in \Omega_T)$.

To formulate conditions for strong ellipticity of the operator $-A_R$, we introduce an auxiliary notation. This notation will also be used in the investigation of the smoothness of the generalized solutions of problem (1.3)–(1.5). Denote by $G$ the additive Abelian group generated by the set $M$ and by $Q_r$ the open connected components of the set

$$Q \setminus \left( \bigcup_{h \in G} (\partial Q + h) \right).$$

**Definition 2.** The sets $Q_r$ are called **subdomains**, and the collection $\mathcal{R}$ of all possible subdomains $Q_r$ is called a **partition** of the set $Q$.

The partition $\mathcal{R}$ is naturally decomposed into disjoint classes as follows. We say that subdomains $Q_{r_1}, Q_{r_2} \in \mathcal{R}$ belong to the same class if there exists an $h \in G$ such that $Q_{r_2} = Q_{r_1} + h$. We denote each of the subdomains $Q_r$ by $Q_{stl}$, where $s$ is the index of the class ($s = 1, 2, \ldots$) and $l$ is the index of the subdomain in the $s$th class. Since the domain $Q$ is bounded, it follows that each class consists of finitely many ($N = N(s)$) subdomains $Q_{s t l}$, and $N(s) \leq ([diam Q] + 1)^n$.

To formulate necessary conditions for strong ellipticity in algebraic form, we introduce the matrices $R_{ij}(x)(x \in Q_{s t l})$ of order $N(s) \times N(s)$ with the entries

$$R_{ij}(x) = \begin{cases} a_{ijk}(x + h_{sk}) & (h = h_{si} - h_{sk} \in M), \\ 0 & (h_{si} - h_{sk} \notin M). \end{cases} \quad (1.6)$$

By [3, Theorem 9.2], if an operator $-A_R$ is strongly elliptic, then the matrices

$$\sum_{i, j=1}^{n} (R_{ij}(x) + R_{ij}(x)) \xi_i \xi_j$$

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