ON THE QUESTION OF EXISTENCE OF MINIMAL NON-FC-GROUPS†
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We recall that a group \(G\) is said to be a \textit{minimal non-FC-group} if \(G\) is not an FC-group, whereas every proper subgroup of \(G\) is an FC-group. In the article [1], there were described minimal non-FC-groups not coincident with their commutator subgroups. It was shown that every perfect minimal non-FC-group must be a quasisimple group or a locally finite \(p\)-group. The investigation by Kuzucuoglu and Phillips in [2] of the situation in [1] have demonstrated that the quasisimple case is impossible in the class of locally finite groups. This implies that if perfect locally finite minimal non-FC-groups exist then they must belong to the class of \(p\)-groups. Some generalization of the results by Kuzucuoglu and Phillips was made in [3] wherein locally graded minimal non-CC-groups were considered.

Leinen and Puglisi [4] observed that perfect locally finite minimal non-FC-groups have nontrivial finitary linear representations. Moreover, the problem of existence of perfect locally finite minimal non-FC-groups was reduced in [4, Theorem 2.4] to a narrower class of finitary linear groups, the subgroups of the so-called MacLane groups \(M(Q, FG(p))\).

In the present article we strengthen the result by Leinen and Puglisi and prove the following

\textbf{Theorem 1.} Every perfect locally finite minimal non-FC-group has nontrivial finitary permutational representation.

Clearly, every nontrivial quotient group of a perfect minimal non-FC-group is a perfect minimal non-FC-group too. Therefore, Theorem 1 reduces the problem of existence of perfect locally finite minimal non-FC-groups to the study of \(p\)-groups of finitary permutations, thereby further restricting the range of choice.

Theorem 1 ensues from the following interesting property of minimal non-FC-groups.

\textbf{Theorem 2.} Let \(G\) be a perfect locally finite minimal non-FC-group and let \(Z(G) = 1\). Then, for every pair of elements \(a\) and \(b\) in \(G\), the element \(b\) does not commute only with finitely many elements conjugate with \(a\) in \(G\).

In light of Theorem 2, we can easily understand the nature of the finitary permutational representation of Theorem 1: this representation is the ordinary action (by conjugation) of \(G\) on every class of conjugate elements. We should note however that by the condition of Theorem 2 the group \(G\) must have the trivial center and so Theorem 2 is not applicable to the situation of Theorem 1 immediately. Before applying it, we must pass to the quotient group \(G/Z(G)\) whose center is trivial since \(G\) is a perfect group.

Theorem 2 follows in turn from Theorem 3 in which the conclusion of Theorem 2 is established under weaker constraints on \(G\) than in Theorem 2. To formulate Theorem 3, we need the following notion: subgroups \(X\) and \(Y\) of some group are \textit{commensurable} if the intersection \(X \cap Y\) has finite index in \(X\) and \(Y\).

\textbf{Theorem 3.} Let \(G\) be an arbitrary group in which

\begin{itemize}
  \item[(1)] the centralizers of all nontrivial elements are commensurable;
  \item[(2)] each element belongs to a normal FC-subgroup.
\end{itemize}

Then, for every pair of elements \(a\) and \(b\) of \(G\), the element \(b\) does not commute only with finitely many elements conjugate with \(a\) in \(G\).

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Let us clarify just now why a perfect locally finite minimal non-FC-group G with trivial center satisfies the conditions of Theorem 3. Commensurability of the centralizers of all noncentral elements of G was demonstrated in [5, Theorem 1], while the validity of the second condition is obvious from the fact that every element of a perfect locally finite p-group belongs to a proper normal subgroup.

We now turn to the proof of Theorem 3. It bases on the following

**Lemma 4.** Let G be an arbitrary group and let a and b be elements of G satisfying the conditions:
1. \(|B : C_B(a)| < \infty\), where \(B = (b^G)\);
2. \(|\{[b, a^x] \mid x \in G\}| < \infty\);
3. \(|C_G([g, a]) : C_G([g, a]) \cap C_G(a)| < \infty\) for every \(g \in B \setminus C_B(a)\);
4. the centralizers of all elements conjugate with a in G are commensurable.

Then the number of elements conjugate with a in G and not commuting with b is finite.

**Proof.** Denote the class \(a^G\) by \(\Omega\) and consider the action (by conjugation) of G on the set \(\Omega\).

Let \(\{\Omega_i \mid i \in I\}\) be the set of all B-orbits of elements in \(\Omega\). Since B is a normal subgroup of G; therefore, \(\{\Omega_i \mid i \in I\}\) forms an imprimitivity G-system in \(\Omega\); moreover, G acts transitively on the set of its components. The first condition of the lemma implies that the B-orbit of a is finite. Hence, all components \(\Omega_i, i \in I\), are finite as well.

Given a component \(\Omega_i\), define the subset \(\Delta_i\) by putting
\[\Delta_i = \Omega_i^{-1} \cdot \Omega_i = \{x^{-1}y \mid x, y \in \Omega_i\}.\]

Observe that the subsets \(\Delta_i, i \in I\), are finite in view of finiteness of the components \(\Omega_i\).

Now, assume that the conclusion of the lemma fails; i.e., assume that the set of elements conjugate with a and not commuting with b is infinite. In this event, there is an infinite set of components \(\Omega_i\), containing at least one element not commuting with b. Denote this set by \(\{\Omega_i \mid i \in I_1\}\). Letting \(K\) stand for the set \(\{[b, a^x] \mid x \in G\}\), we can also define the chosen infinite index set \(I_1\) otherwise:
\[I_1 = \{i \in I \mid \Delta_i \cap K \neq \emptyset\}.\]

In accordance with the second condition of the lemma, \(K\) is an infinite set. Hence, there are an infinite subset \(I_2 \subseteq I_1\) and a nontrivial element \(t \in K\) such that \(t \in \Delta_i\) for all \(i \in I_2\). Now, fix some index \(n\) in \(I_2\) and select elements \(x_i, i \in I_2\), in G for which
\[\Omega_n^{x_i} = \Omega_i.\]

It is clear that these elements satisfy the equalities \(\Delta_n^{x_i} = \Delta_i\) and so \(t^{x_i^{-1}} \in \Delta_n\).

The set \(\Delta_n\) is finite. Hence, there is an infinite subset \(I_3 \subseteq I_2\) such that \(t^{x_i^{-1}} = t^{x_j^{-1}}\) for all \(i, j \in I_3\). In other words, the elements \(x_i^{-1}, i \in I_3\), lie in one left coset of \(C_G(t)\). The membership \(t \in \Delta_n\) means that the element \(t\) may be written as
\[t = a^{-gh} \cdot a^g = g^{-1}[g^h g^{-1}, a]g\]
for some \(a^g\) in \(\Omega_n\) and \(h\) in \(B\). Therefore,
\[C_G(t) = g^{-1}C_G([g^h g^{-1}, a])g.\]

Making use of the third condition of the lemma, we find that the intersection \(C_G([g^h g^{-1}, a]) \cap C_G(a)\) has finite index in \(C_G([g^h g^{-1}, a])\). Therefore, the index of the subgroup \(C_G(t) \cap C_G(a^g)\) in \(C_G(t)\) is finite as well and every left coset of \(C_G(t)\) is covered by finitely many left cosets of \(C_G(a^g)\).

Grounding on the fact that the set \(I_3\) is infinite, we again choose an infinite subset \(I_4\) in it for which all elements of the form \(x_i^{-1}, i \in I_4\), lie in one left coset of \(C_G(a^g)\). Taking the inverse elements, we obtain the set \(\{x_i \mid i \in I_4\}\) which lies in one right coset of \(C_G(a^g)\).

Since all subgroups conjugate with \(C_G(a^g)\) are commensurable (the fourth condition of the lemma), every right coset of \(C_G(a^g)\) is covered by finitely many left cosets of the same subgroup \(C_G(a^g)\). Therefore, in the set \(\{x_i \mid i \in I_4\}\) we may select an infinite subset \(\{x_i \mid i \in I_5\}\) that lies in one left coset of \(C_G(a^g)\). It follows that \((a^g)^{x_i} = (a^g)^{x_j}\) for \(i, j \in I_5\). Now, in view of the membership \(a^g \in \Omega_n\), the components of the form \(\Omega_n^{x_i}\) coincide for all \(i \in I_5\), which contradicts their choice. The contradiction obtained completes the proof of the lemma.