Kloosterman Double Sums

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ABSTRACT. In this paper estimates of incomplete Kloosterman double sums with weights are obtained.

KEY WORDS: Kloosterman double sum, incomplete Kloosterman sum, multiple trigonometric sum.

In [1–5] the author estimated some analogs of incomplete Kloosterman sums and applied the resulting estimates to some arithmetical problems. The method of these papers allows us to estimate of Kloosterman double sums with weights. It should be noted that such sums often occur in applications. Theorems given in the present paper are generalizations of Theorem 4 from [5].

Let us introduce some notation. In what follows, m > 1, m is a positive integer, and a and b are integers; without stating it explicitly, we assume that x and y are positive integers coprime with m; by x* and y* we denote positive integers satisfying the conditions

\[ x^* x \equiv 1 \pmod{m}, \quad y^* y \equiv 1 \pmod{m}, \quad 1 \leq x^*, \quad y^* < m; \]

\( \xi(x) \) and \( \eta(y) \) are arbitrary complex-valued functions of arguments \( x \) and \( y \); \( 0 < X < X_1 \leq 2X \), \( 0 < Y < Y_1 \leq 2Y \); the positive numbers \( \xi, \eta, \xi_0, \eta_0, \xi_1 \) are determined by the relations:

\[ \xi = \max_{X < x \leq X_1} |\xi(x)|; \quad \eta = \max_{Y < y \leq Y_1} |\eta(y)|; \]

\[ \xi_0 = \sum_{X < x \leq X_1} |\xi(x)|; \quad \eta_0 = \sum_{Y < y \leq Y_1} |\eta(y)|; \quad \xi_1 = \sum_{X < x \leq X_1} |\xi(x)|^2. \]

For the proof of the theorems, we need the following result.

**Lemma.** Suppose that \( m > 1, \) \( m \) is an integer, \( k \) is a positive integer, and \( X \) and \( X_1 \) such that

\[ 3 \leq X, \quad k < X < X_1 \leq 2X, \quad k2^{k-1}X^{2k-1} < m. \]

Consider the congruence

\[ x_1^k + \cdots + x_k^k \equiv x_{k+1}^k + \cdots + x_{2k}^k \pmod{m}, \]

where \( X < x_1, \ldots, x_{2k} \leq X_1, \quad (x_1, m) = \cdots = (x_{2k}, m) = 1. \) Then for the number \( I = I_k(X) \) of its solutions the following estimate is valid:

\[ I \leq (2k)^{8k^3} X^k (\log X)^{4k^2}. \]

**Proof.** The proof of this lemma is contained in [3] (also see [5, Theorem 2]). Simple versions of this lemma were obtained in [2]. They are more refined for certain relations between the parameters.

Consider the multiple trigonometric sum \( W = W(a, b), \)

\[ W = \sum_{X < x \leq X_1} \sum_{Y < y \leq Y_1} \xi(x) \eta(y) \exp \left( \frac{2\pi i}{m} (ax^* y^* + bxy) \right). \]

It is natural to call the sum \( W \) a multiple (double) Kloosterman sum with weights. If the product \( (X_1 - X)(Y_1 - Y) \) is less than \( m, \) then \( W \) is called a short or incomplete sum. \( \square \)
Theorem 1. Suppose that \( k \) and \( s \) are positive integers, the numbers \( X, X_1, Y, \) and \( Y_1 \) satisfy the inequalities

\[
3 \leq X, \quad k < X < X_1 \leq 2X, \quad k^{2k-1}X^{2k-1} < m, \\
3 \leq Y, \quad s < Y < Y_1 \leq 2Y, \quad s^{2s-1}Y^{2s-1} < m,
\]

and \( a \) and \( b \) are integers; moreover, \((a, m) = d \geq 1\). Then \( |W| \) satisfies the estimate

\[|W| \leq \xi_0 \eta_0 \Delta,\]

where

\[
\Delta = (2k)^{4s^2/((2s)^{4s^2/k}} (\xi_0^{-1} \xi \sqrt{X})^{1/s} (\eta_0^{-1} \eta \sqrt{Y})^{1/k} (sdm Y)^{1/2s} (\log X)^{2k/s} (\log Y)^{2s/k}.
\]

Proof. We shall follow the arguments of Theorem 3 from [5]. Passing to the inequalities, we obtain

\[
|W| \leq \sum_{X < x \leq X_1} |\xi(x)| \sum_{Y < y \leq Y_1} \eta(y) \exp \left( \frac{2\pi i}{m} (ax^*y^* + bxy) \right).
\]

Let us raise both parts of this inequality to the \( s \)th power and use Hölder’s inequality; we obtain

\[
|W|^s \leq A^{s-1} \sum_{X < x \leq X_1} |\xi(x)| \sum_{Y < y \leq Y_1} \eta(y) \exp \left( \frac{2\pi i}{m} (ax^*y^* + bxy) \right)^s,
\]

where

\[
A = \sum_{X < x \leq X_1} |\xi(x)|.
\]

Using the definition of the numbers \( \xi_0 \) and \( \xi \), we obtain the inequality

\[
|W|^s \leq \xi_0^{s-1} \xi W_1,
\]

where

\[
W_1 = \sum_{X < x \leq X_1} \left| \sum_{Y < y \leq Y_1} \eta(y) \exp \left( \frac{2\pi i}{m} (ax^*y^* + bxy) \right) \right|^s.
\]

Let us raise the sum over \( y \) to the \( s \)th power. To do this, we define the function \( J_s(\lambda, \mu) \) by the relation

\[
J_s(\lambda, \mu) = \sum \eta(y_1) \ldots \eta(y_s),
\]

where the prime on the sum indicates that summation is carried out over collections \( y_1, \ldots, y_s \) satisfying the system of congruences

\[
\begin{aligned}
y_1^* + \cdots + y_s^* &\equiv \lambda \pmod{m}, \\
y_1 + \cdots + y_s &\equiv \mu \pmod{m}, \quad Y < y_1, \ldots, y_s \leq Y_1.
\end{aligned}
\]

We obtain

\[
\left( \sum_{Y < y \leq Y_1} \eta(y) \exp \left( \frac{2\pi i}{m} (ax^*y^* + bxy) \right) \right)^s = \sum_{\lambda = 1}^{m} \sum_{\mu = 1}^{m} J_s(\lambda, \mu) \exp \left( \frac{2\pi i}{m} (ax^* \lambda + b x \mu) \right).
\]

Note that the parameter \( \mu \) in the last relation ranges over the interval

\[
sY < \mu \leq sY_1 < 2sY \leq s^{2s-1}Y^{2s-1} < m.
\]