On the Birational Rigidity of a Series of Multidimensional Fano Varieties

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ABSTRACT. We prove the birational rigidity of algebraic varieties admitting a fibration into a pencil of hypersurfaces of degree $M$ in $\mathbb{P}^M$ having degree 2 over the base.

KEY WORDS: birational rigidity, Fano fibration, maximal singularities method, rational surface.

1. Introduction

One of the most powerful tools in modern birational geometry is the maximal singularities method developed by V. A. Iskovskikh and Yu. I. Manin (see [1]). However, this approach taken in its original form had a very narrow field of applications. Recent technical improvements introduced by A. V. Pukhlikov allow one to widen the field of application substantially; in particular, they make it possible to describe birational automorphisms of varieties of arbitrary dimension.

In [2-4] the birational rigidity of some classes of Fano fibrations satisfying the so-called $K^2$-condition is established. In order to find the boundaries of the area in which the maximal singularities method is applicable, it is useful to study varieties not possessing this property. One such example is considered in [5]. In the present paper we analyze another example of this kind.

Let $V_m$ be a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^M$ given by an equation of the form

$$p_{m,M}(u, v; x_0, \ldots, x_M) = 0,$$

where $m$ is the degree of $p$ over $u$ and $v$, and $M$ is the degree of $p$ over $x_0, \ldots, x_M$. Obviously, there is a projection $\pi: V \to \mathbb{P}^1$ whose fiber is a hypersurface of degree $M$ in $\mathbb{P}^M$. From now on we denote the fiber of $\pi$ by $F$.

It is shown in [2, 3] that for $m \geq 3$ and $M \geq 4$ a general hypersurface $V_m$ is birationally superrigid and possesses a unique Fano fibration structure. On the other hand, $V_1$ is rational for all $M$. The rationality of $V = V_2$ remains an open question.

The variety $V$ is endowed with a second Fano fibration structure. Indeed, let $\sigma: V \to \mathbb{P}^M$ be the projection to the second factor. Then the mapping $\tau: V \dasharrow V$ interchanging the points in fibers of $\sigma$ is well defined outside subset of codimension 2, and it is a birational involution. The second structure is obtained from the first one by applying $\tau$.

Conjecture 1. For $M \geq 4$ and a general variety $V$, the group $\text{Bir} V$ consists of two elements, and it is generated by $\tau$; there are only two fibrations of $V$ with rationally connected fibers.

Conjecture 2. For $M = 3$, the group $\text{Bir} V$ is freely generated by $\tau$ and the group $\text{Bir} F_\eta$, where $F_\eta$ is the fiber of $\pi$ over a general point of $\mathbb{P}^1$. Any pencil of rational surfaces in $V$ can be transformed into $|F|$ by a birational transformation.


2. Statement of the result

Before presenting our central result, let us recall the main notions of the maximal singularities method. The threshold of the canonical adjunction of a linear system $D$ on a Fano fibration $\rho: V \to S$ is the number

$$c(D, W) = \sup \left\{ \frac{b}{a} |a, b \in \mathbb{Z}_+, aD + bK_V| \neq \emptyset \right\}.$$
A Fano fibration $\rho: V \rightarrow S$ is called birationally rigid if for any Fano fibration

$$\rho': V' \rightarrow S', \quad \dim V' = \dim V$$

any birational mapping $\chi: V \rightarrow V'$, and any linear system $|D'|$ on $V'$, which is free in codimension 1, there is a birational automorphism $\chi'$ of $V$ such that

$$c(D, V) \leq c(D', V'), \quad \text{where} \quad |D| = (\chi \circ \chi')^{-1}|D'|.$$ 

Let us prove the following statement.

**Theorem.** If a variety $V$ satisfies the strong regularity conditions formulated below, then it is birationally rigid for $M \geq 6$. For a general $V$, the group $\text{Bir} V$ is isomorphic to $\mathbb{Z}_2$ and is generated by $\tau$.

**Corollaries.** 1) $V$ admits precisely two Fano fibration structures.

2) $V$ is not rational.

Now let us state the generality conditions that we require.

Consider a hypersurface $W$ of degree $M$ in $\mathbb{P}^M$. Let $p$ be an arbitrary point in $W$ and write the equation of $W$ in the affine coordinate system centered at $p$ in the form $q_1 + q_2 + \cdots + q_M = 0$, where $q_i$ is a homogeneous polynomial of degree $i$.

**Definition.** A smooth point $p \in W$ is called strongly regular if the sequence $(q_1, \ldots, q_{M-1})$ is regular, and, besides, the equations $q_2 = \cdots = q_M = 0$ determine a finite set of points in the space $\mathbb{P}^{M-1}$ such that the sum of multiplicities of the points belonging to the hyperplane $q_1 = 0$ does not exceed $M$.

We say that a Fano fibration $V$ is strongly regular if each smooth point of any fiber of the projection $\pi$ is strongly regular, and each singular point of any fiber is regular in the sense of [3]. Counting constants we may show that a general $V$ is strongly regular (see [3, 7]).

### 3. Maximal singularities

Now let $\chi: V \rightarrow (W, D')$ be a birational mapping, $|\chi| = D = \chi^{-1}(D')$, $D = -nK_V + lF$. Suppose that $l < 0$. It is easy to see that $\tau$ acts on $\text{Pic} V$ in the following way:

$$\tau^*K_V = K_V, \quad \tau^*F = \sigma^{-1}(\sigma(F)) - F = MK_V - F,$$

i.e., $(\chi \circ \tau)^{-1}(D') = -nK_V + l(MK_V - F) = -n'K_V + l'F$, where $l' > 0$.

Suppose that $c(D, V) > c(D', W)$. The argument above implies that we can suppose that $l > 0$.

A discrete valuation $\nu$ of the field of rational functions over a variety $V$ is called a maximal singularity if the Noether–Fano–Iskovskikh inequality $e(\nu) = \nu(|\chi|) - nK(\nu, V) > 0$ is valid; here $K(\nu, V)$ is the discrepancy of $\nu$ with respect to $V$.

Similarly to [2, 3], it can be proved that if $c(D, V) > c(D', W)$, then the centers of all maximal singularities are contained in fibers of $\pi$, and there is a finite set of maximal singularities of $\mathcal{M}$ satisfying the following inequality:

$$\sum_{t \in \mathcal{M}} \max_{\nu \in \mathcal{M} \text{ center}(\nu) \subset F_t} \left( \frac{e(\nu)}{\nu(F_t)} \right) > l, \quad \text{where} \quad F_t = \pi^{-1}(t).$$

**Lemma 1.** Let $g$ be a hyperplane section of $F$. Then, if the cycle $NK_V^2 - rg$ is effective, we have the inequality $r < 2N/M$. 

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