Topological Classification of Supertransitive Flows on Closed Nonorientable Surfaces.
Part I. Necessary Conditions for Topological Equivalence

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ABSTRACT. In this paper a topological invariant is introduced for the class of supertransitive flows on closed nonorientable surfaces $M$ of negative Euler characteristic. We describe properties of this invariant and prove that it provides necessary conditions for the topological equivalence of flows belonging to the above-mentioned class of supertransitive flows.

KEY WORDS: dynamical system, nonorientable surface, Fuchsian group, supertransitive flow, homotopic rotation class, equilibrium state, saddle, separatrix.

Introduction

In 1973 S. Kh. Aranson and V. Z. Grines [1] found necessary and sufficient conditions for the topological equivalence of a certain class of flows (dynamical systems with continuous time) on closed orientable surfaces $M$ of negative Euler characteristic; the flows of this class are natural generalizations of irrational windings of a torus and satisfy the following properties: 1) any such flow $f^t$ on $M$ has at least one nonclosed Poisson stable (i.e., self-limit) semitrajectory everywhere dense in $M$; 2) all equilibrium states of $f^t$ are topological saddles of negative index; 3) the flow $f^t$ has no separatrices going from a saddle point to the same or another saddle point. Such flows were subsequently called supertransitive. In particular, it follows that each supertransitive flow $f^t$ has a nonempty finite set of saddles and separatrices. Moreover, any semitrajectory $L$ of $f^t$ whose limit set is not a saddle is a nonclosed Poisson stable semitrajectory everywhere dense in $M$. In [2] Aranson also solved the admissibility and realization problems for the complete topological invariant found in [1], i.e., the homotopic rotation class.

The importance of the study of such flows is related to the fact that, on the one hand, they are the simplest objects closely connected with the topology and geometry of the underlying surface and, on the other hand, integral curves of such flows can serve as invariant manifolds of cascades (dynamical systems with discrete time) like those generated by pseudo-Anosov homeomorphisms, which were introduced by W. Thurston and played an important role in the homotopic and topological classification of homeomorphisms on surfaces.

Supertransitive flows on nonorientable surfaces $M$ are defined quite similarly. The study of supertransitive flows on closed nonorientable surfaces remained outside the mainstream for long because of the specific properties of the fundamental group of such surfaces as compared to those of closed orientable surfaces; however, isolated results in this area were obtained. For example, in [3] it was shown that there are no supertransitive flows on closed nonorientable surfaces of genus 3, and in [4, 5] it was proved that such flows exist on closed nonorientable surfaces of genus more than or equal to 4. Note that supertransitive flows do not exist on closed nonorientable surfaces of nonnegative Euler characteristic, i.e., there are no such flows on the projective plane and on the Klein bottle. This follows from the fact that the set of equilibrium states of each supertransitive flow is nonempty, and the equilibrium states themselves are topological saddles of negative index; hence the Euler characteristic of the underlying closed surface must be negative, since the sum of the indices of all equilibrium states is equal to the Euler characteristic. For the same reasons, there are no supertransitive flows on the sphere and on the torus. As for other closed orientable surfaces, supertransitive flows on them exist; they can be easily obtained either with the help...
of the operation of branching over certain points of the torus on which an irrational winding is given, or
by a direct construction (on this subject, see the monographs of S. Kh. Aranson, E. V. Zhuzhoma, and
G. R. Belitskii [6], and of I. V. Nikolaev and E. V. Zhuzhoma [7]).

In 1998 Aranson, Zhuzhoma, and Tel'nykh [8] announced necessary and sufficient conditions for super-
transitive flows on closed nonorientable surfaces of genus more than or equal to 4 to be topologically
equivalent, and in the deposited manuscript of Aranson and Tel'nykh [9] some aspects of the proofs are
presented.

In the present paper a special covering group for closed nonorientable surfaces $M$ of negative Euler
characteristic is constructed and a topological invariant of supertransitive flows on $M$, i.e., the orbit of
the homotopic rotation class, is introduced; further, a theorem on necessary conditions for the topological
equivalence of such flows is proved.

1. Properties of the covering group

In what follows, $M$ is a nonorientable surface of genus $p \geq 4$. By $\overline{M}$ denote the closed orientable
surface of genus $\overline{p} = p - 1$ which is a two-fold covering of $M$, and by $\pi_1$ denote the projection of $\overline{M}$
onto $M$. By the uniformization theorem, $\overline{M}$ can be represented as a quotient space $\overline{M}/\Gamma$, where $\overline{M}$ is the
surface $x^2 + y^2 < 1$ in the complex $z$-plane ($z = x + iy$) and $\Gamma$ is a discrete group of transformations of $\overline{M}$;
the natural projection $\pi_2: \overline{M} \rightarrow M$ is a universal unramified covering of $\overline{M}$. The circle $E: x^2 + y^2 = 1$
is referred to as the absolute. The surface $M$ is represented as the quotient space $M/G$, where $G$ is
the covering group for $M$, and the covering $\pi = \pi_1 \circ \pi_2: \overline{M} \rightarrow M$ is the universal unramified covering;
besides, $\Gamma$ is a subgroup of $G$ of index 2.

Let us introduce a metric of constant negative curvature on $\overline{M}$. Then the geodesics with respect to
this metric are the circular arcs orthogonal to the absolute with endpoints on $E$. By the projections $\pi_2$
and $\pi$, this metric induces metrics of constant negative curvature on $\overline{M}$ and on $M$, respectively. Each
nonidentity transformation in $\Gamma$ is a hyperbolic element of the first kind, i.e., it has the form

$$\gamma(z) = \frac{az + \beta}{\beta z + \alpha}, \quad \alpha \overline{\alpha} - \beta \overline{\beta} = 1, \quad |\alpha + \overline{\alpha}| > 2.$$ 

Here the bar denotes complex conjugation. Each hyperbolic element of the first kind preserves orientation
and has exactly two fixed points, i.e., an attracting one $\sigma^+ \in E$ and a repelling one $\sigma^- \in E$.

We define a canonical hyperbolic element of the first kind as a transformation of the form

$$\gamma_0(z) = \frac{\alpha_0 z + \beta_0}{\beta_0 z + \alpha_0}, \quad \alpha_0 = -\frac{1 + K}{2\sqrt{K}}, \quad \beta_0 = \frac{1 - K}{2\sqrt{K}}, \quad K > 1.$$ 

The fixed points of $\gamma_0$ are the attracting one $\sigma_0^+ = 1$ and the repelling one $\sigma_0^- = -1$.

A hyperbolic element of the second kind is a transformation of the form

$$g(z) = \frac{\alpha \overline{z} + \beta}{\beta \overline{z} + \alpha}, \quad \alpha \overline{\alpha} - \beta \overline{\beta} = 1, \quad \text{Re } \beta \neq 0.$$ 

Each hyperbolic element of the second kind changes orientation and has exactly two fixed points, namely,
an attracting one $\sigma^+ \in E$ and a repelling one $\sigma^- \in E$. Direct verification shows that $g^2$ is a hyperbolic
element of the first kind.

We define a canonical hyperbolic element of the second kind as an element of the form $g_0(z) = \gamma_0(z)$,
where $\gamma_0(z)$ is a canonical hyperbolic element of the first kind. The fixed points of $g_0$ are the attracting
one $\sigma_0^+ = 1$ and the repelling one $\sigma_0^- = -1$; besides, we have $g_0^2 = \gamma_0^2 \in G$. In particular, any element
of the form $\gamma g_0 \gamma^{-1}$, where $g_0 \in G$ is a canonical hyperbolic element of the second kind and $\gamma \in \Gamma$, is a
hyperbolic element of the second kind.

The elements of the Fuchsian group, which transform the disk $\overline{M} = \{z| < 1$ onto itself and leave the
circle $|z| = 1$ invariant, can be of the following types, except for the hyperbolic elements of the first

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