1. Introduction

It was shown by Kegel [1] that every finite group \( G = AB = AC = BC \), factorized by two nilpotent subgroups \( A \) and \( B \) and a supersoluble subgroup \( C \), is supersoluble too. The authors extended this result in [2] to the case of a trifactorized soluble-by-finite group \( G = AB = AC = BC \) with finite abelian section rank, proving that if \( A \) and \( B \) are nilpotent and \( C \) is locally supersoluble then \( G \) itself is locally supersoluble. Here a group \( G \) is said to have finite abelian section rank if it lacks infinite abelian sections of prime exponent. Even in the case of finite groups, it is clearly not enough to assume that the subgroups \( A, B, \) and \( C \) all are supersoluble. In fact, there exists a finite nonsupersoluble group \( G = AB = AC = BC \) written as the product of two supersoluble normal subgroups \( A \) and \( B \) and a nilpotent subgroup \( C \) (see, for instance, [13, p. 152]). On the other hand, Baer [4] proved that, if \( G \) is a finite group with nilpotent commutator subgroup and \( H \) and \( K \) are supersoluble normal subgroups of \( G \), then the product \( HK \) is supersoluble. This result suggests that the behavior of the commutator subgroup is the main obstacle in studying the groups factorized by supersoluble subgroups. In fact, we prove in this article that, if \( G = AB = AC = BC \) is a group with finite abelian section rank factorized by three locally supersoluble subgroups \( A, B, \) and \( C \) and the commutator subgroup \( G' \) of \( G \) is locally nilpotent, then \( G \) is locally supersoluble. An analogous result holds for groups with finite abelian section rank having a triple factorization by locally nilpotent subgroups. These results are proved in Section 2, where an extension of Baer's theorem to infinite groups can also be found. Finally, in Section 3 we consider groups with a triple factorization by subgroups having (generalized) nilpotent commutator subgroups.

Most of our notation is standard and can be found in [3]. We refer to [5] for the main properties of factorized groups.

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2. Groups with a Supersoluble Triple Factorization

Our first result generalizes Baer's theorem on products of supersoluble normal subgroups to the case of infinite groups. Recall that a group \( G \) is FC-hypercentral if it has an ascending normal series

\[ 1 = G_0 \leq G_1 \leq \cdots \leq G_\tau = G \]

such that every element of \( G_{\alpha+1}/G_\alpha \) has finitely many conjugates in \( G/G_\alpha \) for each ordinal \( \alpha < \tau \).

Lemma 2.1. Let \( G \) be a group with locally nilpotent commutator subgroup and let \( H \) and \( K \) be normal subgroups of \( G \).

(a) If \( H \) and \( K \) are supersoluble then \( HK \) is supersoluble.

(b) If \( H \) and \( K \) are locally supersoluble then \( HK \) is locally supersoluble.

(c) If \( H \) and \( K \) are hypercyclic then \( HK \) is hypercyclic.

Proof. (a) Clearly, the group \( L = HK \) is polycyclic and all its finite homomorphic images are supersoluble (see [4, p. 186]). It follows that \( L \) itself is supersoluble by a result of Baer (see [6]).
Let $E$ be a finitely generated subgroup of $L = HK$. Then there exist finitely generated subgroups $H_0$ of $H$ and $K_0$ of $K$ such that $E$ is contained in $M = \langle H_0, K_0 \rangle$, and clearly $M = (H \cap M)(K \cap M)$. Since $L$ is locally polycyclic (see [7, Part 1, Theorem 2.31]), the group $M$ is polycyclic, so that $H \cap M$ and $K \cap M$ are supersoluble, and so $M$ is supersoluble by (a). Therefore, $L$ is locally supersoluble.

(c) Let $L = HK$. As the hypotheses are inherited by homomorphic images, it is enough to show that, if $L \neq 1$, then $L$ contains a cyclic nontrivial normal subgroup. The subgroups $H$ and $K$ are FC-hypercentral, so that $L$ is FC-hypercentral too (see [7, Part 1, p. 130]). Let $x \neq 1$ be an element of $L$ having only finitely many conjugates. Then $\langle x \rangle^L$ is finitely generated, and $C = C_L(\langle x \rangle^L)$ is a normal subgroup of finite index in $L$. Therefore, there exists a finitely generated subgroup $E$ of $L$, containing $\langle x \rangle^L$, such that $L = CE$. Since $L$ is locally supersoluble by (b), the subgroup $E$ is supersoluble, and so it contains a cyclic nontrivial normal subgroup $N$ such that $N \leq \langle x \rangle^L$. Clearly, $C \leq C_L(N)$, and hence $N$ is normal in $L$. 

Lemma 2.2. Let $G$ be a group with locally nilpotent commutator subgroup and let $H$ be a locally supersoluble ascendant subgroup of $G$. Then the normal closure $H^G$ is locally supersoluble.

Proof. Let $$H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_\tau = G$$be an ascending series with smallest ordinal type $\tau$. By induction on $\tau$ we may suppose that the subgroup $H_\alpha^H$ is locally supersoluble for each ordinal $\alpha < \tau$. If $\tau$ is a limit ordinal, then $$G = \bigcup_{\alpha < \tau} H_\alpha$$and hence $$H^G = \bigcup_{\alpha < \tau} H_\alpha^H$$is obviously locally supersoluble. Suppose that $\tau$ is not a limit ordinal, and put $K = H_{\tau-1}$. As $H^K$ is locally supersoluble, on replacing $H$ by $H^K$ we can assume without loss of generality that $H$ is a normal subgroup of $K$. Then $H_x^K$ is a locally supersoluble normal subgroup of $K$ for every element $x$ of $G$. From Lemma 2.1 it follows now that $H^G$ is locally supersoluble. 

Theorem 2.3. Let $G$ be a group with locally nilpotent commutator subgroup and let $H$ and $K$ be ascendant subgroups of $G$.

(a) If $H$ and $K$ are supersoluble then $\langle H, K \rangle$ is supersoluble.

(b) If $H$ and $K$ are locally supersoluble then $\langle H, K \rangle$ is locally supersoluble.

Proof. Suppose first that $H$ and $K$ of $G$ are locally supersoluble ascendant subgroups of $G$. Then the normal subgroups $H^K$ and $K^K$ are locally supersoluble by Lemma 2.2, and Lemma 2.1 implies that $H^K K^K$ is locally supersoluble. In particular, $\langle H, K \rangle$ is a locally supersoluble subgroup of $G$. Assume now that $H$ and $K$ are supersoluble. Then the locally supersoluble subgroup $\langle H, K \rangle$ is finitely generated and hence supersoluble. 

Note that a result similar to statements (a) and (b) of Theorem 2.3 does not hold for the join of hypercyclic ascendant subgroups of a group. In fact, there exists a locally finite 2-group not hypercentral but generated by two abelian subnormal subgroups (see [8, p. 22]).

A group $G$ is called parasoluble if it has a normal series of finite length $$1 = G_0 \leq G_1 \leq \cdots \leq G_t = G$$such that for every $i \leq t - 1$ the group $G_{i+1}/G_i$ is abelian and all its subgroups are normal in $G/G_i$. Clearly, a group is supersoluble if and only if it is parasoluble and finitely generated. Moreover, the commutator subgroup of a parasoluble group is nilpotent. The following result on products of parasoluble normal subgroups in particular provides an alternative proof of Baer's theorem for finite groups.