**GQ(4, 2)-Extensions, Strongly Regular Case**

A. A. Makhnev

**ABSTRACT.** A strongly regular locally GQ(4, 2)-graph is a graph with parameters either (126, 45, 12, 8) or (190, 45, 12, 10). The existence and the uniqueness of the corresponding locally GQ(4, 2)-graph in the first case are well known. We prove that the GQ(4, 2)-hyperoval on ten vertices either is the Petersen graph, or is the Möbius 5-prism, or consists of two (2, 3)-subgraphs connected by three edges. We obtain homogeneous GQ(4, 2)-solutions with a strongly regular point graph; in particular, this implies the negative answer to the question of F. Buekenhout concerning the existence of a locally GQ(4, 2)-graph with the parameters (190, 45, 12, 10).

**KEY WORDS:** finite geometry, point graph, strongly regular graph, hyperoval, Petersen graph, prism, Möbius prism.

A geometry $G$ of rank 2 is an incidence system with the set of points $\mathcal{P}$ and the set of blocks $\mathcal{B}$ (containing no repeating blocks). Each block can be identified with the set of points incident to this block; under this identification the incidence relation coincides with the usual inclusion. Two distinct points from $\mathcal{P}$ are called collinear iff there is a block containing both of them. The residue $G_a$ of a geometry $G$ at a point $a$ is the geometry on the set of points $\mathcal{P}_a$ collinear to $a$ with the set of blocks $B_a = \{B \setminus \{a\} | a \in B \in \mathcal{B}\}$.

Let $\mathcal{F}$ be a family of rank 2 geometries. A geometry $G$ of rank 2 is called an $\mathcal{F}$-extension iff each residue $G_a$ belongs to $\mathcal{F}$. The point graph $\Gamma = \Gamma(G)$ is the graph on the set of vertices $\mathcal{P}$ such that any two vertices are adjacent iff they are collinear. The diameter of a geometry, the connectedness, the distance between two points and so on coincide with the usual notions in the graph $\Gamma$. For a vertex $a$ of $\Gamma$ denote by $[a]$ the neighborhood of $a$, i.e., the subgraph induced by $\Gamma$ on the set of all vertices adjacent to $a$. We set $a^+ = \{a\} \cup [a]$. For a subgraph $\Delta$ of $\Gamma$, denote by $K_i(\Delta)$ the set of vertices out of $\Delta$ that are adjacent to $i$ vertices of $\Delta$. The complete bipartite graph with parts $m, n$ will be called the $(m, n)$-graph.

Take $a \in \mathcal{P}$, $B \in \mathcal{B}$ such that $a \notin B$ (the pair $(a, B)$ is called an antiflag). Denote the number of points from $B$ that are collinear to $a$ by $f(a, B)$. A geometry $G$ is called $\varphi$-homogeneous iff $f(a, B)$ equals either 0, or $\varphi$ for any antiflag $(a, B)$; $G$ is called strongly $\varphi$-homogeneous iff $f(a, B) = \varphi$ for any antiflag $(a, B)$. If the intersection of any two blocks from $B$ contains not more than one point, then the blocks are called the lines.

A geometry $G$ is called an $\alpha$-partial geometry of order $(s, t)$ iff each line contains $s + 1$ points, each point belongs to $t + 1$ lines, and $G$ is strongly $\alpha$-homogeneous (notation $pG_\alpha(s, t)$). A $pG_1(s, t)$-geometry is called the generalized quadrangle of order $(s, t)$ and denoted by $GQ(s, t)$. Further, $pG_{s+1}(s, t)$ is a Steiner 2-scheme and $pG_t(s, t)$ is a net.

A geometry $EpG_\alpha$ is called an extension of $\alpha$-partial geometries iff it is connected and each its residue has the form $pG_\alpha(s, t)$ for some $s, t$. The order $(s, t)$ is independent of the chosen residue, because of connectedness, and the geometry is denoted by $EpG_\alpha(s, t)$. A geometry $EpG_\alpha(s, t)$ is called triangular iff it is $(\alpha + 1)$-homogeneous.

For a graph $\Gamma$ and vertices $a, b \in \Gamma$, the number of vertices in $[a] \cap [b]$ is denoted by $\mu(a, b)$ (resp., $\lambda(a, b)$) if the distance between $a$ and $b$ is 2 (resp., if $a$ and $b$ are adjacent). Further, the subgraph induced by $[a] \cap [b]$ is called a $\mu$-subgraph (resp., a $\lambda$-subgraph). A graph $\Gamma$ is totally regular with parameters $(v, k, \lambda, \mu)$ iff it has $v$ vertices, is regular of valency $k$, any one of its edges belongs to...
a $\lambda$-triangle, and any pair of its vertices at distance 2 has $\mu$ common neighbors. A totally regular graph of diameter 2 is called strongly regular.

A subgraph of a point $GQ(s, t)$-graph is called a hyperoval iff it is a regular graph of valency $t - 1$ with an even number of vertices and has no triangles. Note that only hyperovals can be $\mu$-subgraphs of triangular $GQ(s, t)$-extensions. Triangular extensions of generalized quadrangles with $s = 2$ were described and the list of all strongly regular locally $GQ(s, t)$-graphs for $s = 3, 4$ and classical $GQ$ was obtained in [2]. The existence of locally $GQ(4, 2)$-graphs with parameters $(190, 45, 12, 10)$ is mentioned in [2] as an unsolved problem. The paper [2] completed the classification of triangular extensions of generalized quadrangles with $s = 3$ (locally $GQ(3, t)$-graphs) started in the papers [3] by the present author and [4, 5] by D. Pasechnik. An important part of this research is the classification of hyperovals in $GQ(3, t)$.

The main goal of the present paper is the classification of homogeneous $EGQ(4, 2)$ with a strongly regular point graph. As a consequence, we prove the nonexistence of totally regular locally $GQ(4, 2)$-graphs with $\mu = 10$.

It is well known [6] that there is a unique $GQ(4, 2)$ denoted by $H_3(4)$. Let $\Gamma$ be the point graph for $H_3(4)$. Then $\Gamma$ is a strongly regular graph with parameters $(45, 12, 3, 3)$ such that each pair of its points is regular. Thus, if points $a, b$ are not adjacent, then $\Gamma$ contains a unique point $c$ distinct from $a$ and $b$ such that it is adjacent to all three points constituting $[a] \cap [b]$.

**Theorem.** There are no totally regular locally $GQ(4, 2)$-graphs with $\mu = 10$.

**Corollary.** Let $G$ be a homogeneous $GQ(4, 2)$-extension with strongly regular point graph $\Gamma$. Then either $G$ is a one-point extension (in particular, a $2 - (46, 6, 3)$-scheme), or $\Gamma$ is the unique locally $GQ(4, 2)$-graph with parameters $(126, 45, 12, 8)$.

The existence of a one-point extension $GQ(4, 2)$ is proved in [7]. The existence of a locally $GQ(4, 2)$-graph with parameters $(126, 45, 12, 8)$ is well known, and the uniqueness is proved in [8]. This is the graph on the set of vertices of norm 1 in the 6-dimensional orthogonal space over $GF(3)$ of type "+" with the adjacency relation given by orthogonality; its automorphism group is $O^+(3)$. It is interesting that this graph admits a 3-covering, which is a $GQ(4, 2)$-graph of diameter 4. Note that in [9] Pasechnik found one more (not totally regular) locally $GQ(4, 2)$-graph. Let us start the proof of the theorem.

**Lemma 1.** Let $\Lambda$ be a hyperoval of $GQ(4, 2)$ on ten vertices, $K_i = K_i(\Lambda)$, $x_i = |K_i|$. Then one of the following statements is true:

1) $x_0 = x_4 = 0$, $x_2 = 30$, $x_6 = 5$, $\Lambda$ is the Petersen graph; the Petersen graph has precisely five maximal matchings containing three edges each, and it is a subgraph of the neighborhood of precisely one of the points from $K_6$;
2) $x_0 = 4$, $x_2 = 18$, $x_4 = 12$, $x_6 = 1$, $\Lambda$ consists of two $(2, 3)$-subgraphs connected by three edges from the neighborhood of a point from $K_6$;
3) $x_0 = 5$, $x_2 = x_4 = 15$, $x_6 = 0$, $\Lambda$ is the Möbius 5-prism (see Fig. 1), and each point of $K_4$ is incident either to a pair of vertical edges, or to a pair of edges belonging to the bases of the prism incident to opposite points in some quadrangle from $\Lambda$.

**Proof.** Counting the number of edges connecting $\Lambda$ and $\Gamma - \Lambda$ we conclude that the coefficients $x_i$ satisfy the following system of equations:

$$x_0 + x_2 + x_4 + x_6 = 30, \quad 2x_2 + 4x_4 + 6x_6 = 90.$$ (*)&

If a line does (not) intersect $\Lambda$, then we call it the secant (the external). Let us show that if a secant line $L$ contains a point from each of $K_i, K_j, K_r$, then $i + j + r = 10$. Indeed, $L$ contains two points from $\Lambda$, and each of these points is adjacent to two points from $\Lambda - L$. Further, each point from $\Lambda - L$ is adjacent to a unique point from $L$, therefore $(i - 2) + (j - 2) + (r - 2) = (|\Lambda| - 2) - 4$.

If a secant line $L$ contains a point $a$ from $K_6$, then $L$ contains two points from $K_2$, and each point from $[a] \cap \Lambda$ is adjacent to two points from $\Lambda - [a]$ (all in all, there are twelve edges of this type). Hence, each point from $\Lambda - [a]$ is adjacent to three points from $\Lambda \cap [a]$, and $\Lambda - [a]$ is a coclique. On the other