Renormalization of Lee-Type Models in Spaces of Arbitrary Dimension

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The renormalization problem arises in quantum field theory. It consists in assigning a mathematical meaning to certain diverging expressions [1]. For local relativistic theories in four-dimension space-time, this problem was solved only in the framework of perturbation theory. In this connection, for a detailed analysis of the renormalization problem, it seems to be relevant to study simple solvable models [2, 3]. One can expect that the renormalization methods applicable to solvable models will also be applicable to local field theories.

In this note we consider the Lee model [3–5] arising in nuclear physics. In the one and two-dimensional cases, this model can be constructed by means of the method of self-adjoint extensions of operators in Hilbert space [2, 6], and in three-dimensional case this can be done by a similar method for a space with indefinite metric [3]. For higher dimensions, a rigorous construction of the model by the methods given in [3, 7] is impossible. By the classification from [6], the bilinear form corresponding to the interaction Hamiltonian of the model is \((d - 1)\)-singular in the \(d\)-dimensional case; the methods developed in [6] can be applied for \(s\)-singular bilinear forms for \(s = 0, 1\).

To construct a renormalized model, in this note we use the approach developed in [8]. We consider a regularized equation which depends on a small parameter instead of the ill-posed parent equation, and we add extra terms if necessary. Then we impose some assumptions on the dependence of the initial condition and coefficients of the regularized equation on the small parameter. Moreover, the assumption on the initial condition must be invariant with respect to temporal evolution. The class of initial conditions satisfying the imposed assumptions plays the role of the state space.

The specific form of the conditions imposed on the initial data can be derived, for example, by the Bogoliubov procedure [1] based on the smooth switching (on and off) of the interaction.

In the Lee model, the states are determined by the sets \((c, \varphi(k))\) consisting of the probability amplitude \(c\) (a complex number) of the existence of only one particle in the system and a complex function \(\varphi(k)\), which is the amplitude of the existence of two particles with momenta \(k\) and \(-k\). To describe the evolution of the system, one considers the system of equations

\[
\frac{dc}{dt} = g \int dk \xi(k)\varphi(k), \quad i\frac{d\varphi}{dt} = \omega \varphi + gc\xi,
\]

where \(\omega\) is the operator of multiplication by the function \(\omega(k) = \sqrt{k^2 + \mu^2}\), \(g, \mu \in \mathbb{R}\), \(k \in \mathbb{R}^d\), \(\xi(k) = \omega(k)^{-1/2}\). The argument \(t\) is skipped in the notation of the functions \(\varphi\) and \(c\). System (1) is ill-posed because of the ultraviolet divergence of the integral, which occurs since the decrease rate of the function \(\xi(k)\) as \(k \to \infty\) is insufficient.

To make system (1) mathematically rigorous, let us consider some regularization and renormalization procedures. Let \(a\) be a regularization parameter, which tends to zero in what follows.

We replace the function \(\xi\) by the function \(\xi^a\), which decreases rapidly as \(k \to \infty\) and converges to \(\xi\) as \(a \to 0\). Moreover, we add the extra term \(\sum_{l=0}^{m} i^l z^{(l)} c^{(l)}\), where \(c^{(l)} = \frac{d^l}{dt^l} c\), to the right-hand side of the first equation of system (1):

\[
\frac{dc}{dt} = g(\xi^a, \varphi^a) + \sum_{l=0}^{m} i^l z^{(a)} c^{(a)(l)} , \quad i\frac{d\varphi^a}{dt} = \omega \varphi^a + gc^a \xi^a ,
\]


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Let us find the dependence of the extra terms $z_t$ on the regularization parameter $a$ so that system (2) will make sense in the limit as $a \to 0$. Let us study system (2) with the coefficient $g = g_t$ depending on $t$ and vanishing outside of some interval. After the change of variables $\varphi_t = e^{-i\omega t} \phi_t$, system (2) can be rewritten as follows:

$$
\begin{align*}
\phi_t &= \phi_{-\infty}^a - i \sum_{l=0}^{m-1} (-1)^l (g_t c_{l+1})^{(n)} \int_{-\infty}^{t} e^{i\omega r} (g_t c_r^a)^{(m)} \xi^a, \\
g_t (\xi^a, e^{-i\omega t} \phi_{-\infty}) + \sum_{l=0}^{m} i^l z_t^a c_t^{(l)} + i^{m-1} g_t \int_{-\infty}^{t} e^{-i\omega (t-r)} (g_t c_r^a)^{(m)} \xi^a = 0,
\end{align*}
$$

where the $z_t^a$ stand for coefficients determined by the relation

$$
\sum_{l=0}^{m} i^l z_t^a \frac{d^l}{dt^l} - \frac{i}{d} \sum_{l=0}^{m-1} i^l g(t) \left( \xi^a, \frac{1}{\omega^{l+1}} \xi^a \right) \frac{d^l}{dt^l} g(t) = \sum_{l=0}^{m} i^l z_t^a \frac{d^l}{dt^l}.
$$

\begin{align*}
\phi_t^a &= \phi_{-\infty}^a - i \sum_{l=0}^{m-1} (-1)^l (g_t c_{l+1})^{(n)} \int_{-\infty}^{t} e^{i\omega r} (g_t c_r^a)^{(m)} \xi^a, \\
g_t (\xi^a, e^{-i\omega t} \phi_{-\infty}) + \sum_{l=0}^{m} i^l z_t^a c_t^{(l)} + i^{m-1} g_t \int_{-\infty}^{t} e^{-i\omega (t-r)} (g_t c_r^a)^{(m)} \xi^a = 0,
\end{align*}

System (3) is regular as $a \to 0$ if the coefficients $z_t^a$ have limits as $a \to 0$.

**Theorem 1.** Assume that $z_m^a \geq C_1 > 0$, $\omega \geq C_2 > 0$, the function $\phi_{-\infty}^a(k)$ is independent of the parameter $a$ and belongs to the Schwartz space, $(\xi, \omega^{-m} \xi) < \infty$, and $\| \xi^a / \omega^{m/2} - \xi / \omega^{m/2} \|_{L^2} \to 0$. Then

$$
c^a(l) \to c(l), \quad l = 0, \ldots, m-1,
$$

as $a \to 0$, and the function $\varphi^a$ determined by the relation

$$
\varphi^a = - \sum_{l=0}^{m-1} i^l (g c^a)^{(l)} \frac{1}{\omega^{l+1}} \xi^a + \varphi^a,
$$

possesses the property

$$
\| \omega^{m/2} \varphi^a - \omega^{m/2} \varphi \|_{L^2} \to 0.
$$

In Eqs. (4), (6), the convergence is uniform on the interval $t \in (t_1, t_2)$.

It turns out that the conditions imposed in Theorem 1 on the solution of system (2) are invariant with respect to temporal evolution.

**Theorem 2.** Assume that under the assumptions of Theorem 1, the initial condition for system (2) possesses properties (4) and (6). Then the solution of system (2) possesses the same properties.

**Remarks.** 1. The assertions similar to the assertions of Theorems 1 and 2, are true under the weaker assumptions $(\xi, \omega^{-(m+1)} \xi) < \infty$, $\| \xi^a / \omega^{(m+1)/2} - \xi / \omega^{(m+1)/2} \|_{L^2} \to 0$, if the kernel of the integral equation satisfies the polarity condition:

$$
\left( \xi^a, \frac{e^{-i\omega t}}{\omega^{(m+1)} \xi^a} \right) t^a \leq C_3, \quad 0 < \alpha < 1.
$$

In that case, property (6) can be replaced by

$$
\| \omega^{(m+1)/2} \varphi^a - \omega^{(m+1)/2} \varphi \|_{L^2} \to 0.
$$

2. If $z_m = 0$ in the assumptions of the previous Remark, an assertion similar to Theorem 2 is true for $g \neq 0$. The proof of this assertion follows from the general theory of Volterra equations of the first order with singular kernel [9].