Almost Semirecursive Sets

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ABSTRACT. Let \( A \) be a subset of \( \mathbb{N} = \{0, 1, 2, \ldots \} \), and let \( a \notin A \). The set \( A \) is said to be almost semirecursive, if there is a two-place general recursive function \( f \) such that
\[
f(z, y) \in \{z, y, a\} \land (\{z, y\} \subseteq A \iff f(z, y) \in A)
\]
for all \( z, y \in \mathbb{N} \). Among other facts, it is proved that if \( A \) and \( \mathbb{N} \setminus A \) are almost semirecursive sets, then \( A \) is a semirecursive set, and that there exists a wsr*-set that is neither a wsr-nor an almost semirecursive set.

KEY WORDS: semirecursive set, simple set, immune set, retraceable set, recursive enumerability.

Suppose that \( \mathbb{N} = \{0, 1, 2, \ldots \} \), \( A \subseteq \mathbb{N} \), and \( a \notin A \). We say that \( A \) is an almost semirecursive set, if there exists a two-place general recursive function \( f \) such that
\[
f(z, y) \in \{z, y, a\} \land (z \in A \land y \in A \iff f(z, y) \in A)
\]
for all \( z, y \in \mathbb{N} \).

Replacing \( \{z, y, a\} \) in (1) by \( \{z, y\} \), we obtain one of the possible definitions of a semirecursive set [1], which plays an important part in the theory of recursion. The aim of this note is to elucidate some properties of these sets and compare them with other classes of sets close to semirecursive sets.

A set \( A \subseteq \mathbb{N} \) is called weakly semirecursive (wsr-) [2] if there exists a two-place partially recursive function \( f \) such that for all \( z, y \in \mathbb{N} \),
\[
((z \notin A \land y \notin A) \lor (z \notin A \land y \in A)) \implies (f(z, y) \downarrow \land f(z, y) \in \{z, y\} \cap A);
\]
\( A \) is semirecursively enumerable (sre-) if
\[
(z \in A \lor y \in A) \implies (f(z, y) \downarrow \land f(z, y) \in \{z, y\} \cap A);
\]
and \( A \) is a wsr*-set [3] if
\[
\begin{align*}
(a) & \; f(z, y) \downarrow \implies f(z, y) \in \{z, y\}; \\
(b) & \; z \in A \land y \in A \implies f(z, y) \downarrow; \\
(c) & \; f(z, y) \downarrow \land f(z, y) \in A \implies z \in A \land y \in A.
\end{align*}
\]

Note that earlier [4], wsr*-sets were also called weakly semirecursive. It is clear that semirecursive sets are sre-sets, and the latter are wsr-sets. Also, it can be easily shown [2] that a set \( A \) is semirecursive if and only if \( \overline{A} = \mathbb{N} \setminus A \) and \( A \) are sre-sets, and each recursively enumerable set is an sre-set. It follows that if \( A \) is a recursively enumerable nonsemirecursive set, then \( \overline{A} \) is a wsr-, but not a wsr*-set [2].

On the other hand, in [3], an example of a wsr*-, but not a wsr-set is given; also, it is shown that \( A \) is an sre-set if and only if \( A \) is both a wsr-and a wsr*-set. As regards almost semirecursive sets, they are wsr*-sets.

**Proposition 1.** Each nonrecursive tt-degree contains an almost semirecursive set that is not weakly semirecursive.
Proof. Let us introduce the lexicographic order in the set $\Omega$ of all finite zero-one sequences:

$$0 < 1 < 00 < 01 < 10 < 11 < 000 < 010 < 011 < 100 < \cdots,$$

and then enumerate them by the integers $0, 1, 2, \ldots$ in ascending order. This yields an effective one-to-one correspondence between elements of $\Omega$ and their numbers. Let

$$\theta_n = \begin{cases} 
1 & \text{if } n \in B, \\
0 & \text{if } n \notin B,
\end{cases} \quad B \subseteq \mathbb{N},$$

and let $A = \{\theta_0, \theta_0 \theta_1, \theta_0 \theta_1 \theta_2, \ldots\}$. We clearly have $A =_{\mathit{tt}} B$. We shall write $\alpha < \beta$ if $\alpha$ is a beginning of $\beta$. Let

$$g(\alpha, \beta) = \begin{cases} 
\alpha & \text{if } \beta < \alpha, \\
\beta & \text{if } \alpha < \beta, \\
\gamma & \text{otherwise},
\end{cases}$$

where $\gamma$ is a fixed element from $\Omega \setminus A$. It is readily seen that $A$ turns out to be an almost semirecursive set by the general recursive function $g$. Suppose that $A$ is a nonrecursive set, and for $A$ there exists a partially recursive function $f$ satisfying (2). But then for any $\alpha \in A$ we have

$$f(\alpha_0, \alpha_1) \downarrow \land (\alpha_1 \in A \iff f(\alpha_0, \alpha_1) = \alpha_1).$$

This contradicts the fact that $A$ is a nonrecursive set. \square

**Theorem 1.** There exists a wsr*-set which is neither a wsr-nor an almost semirecursive set.

Proof. Let $B$ be a hypersimple set with retraceable complement. We can find a general recursive function $h$ representing $B$ such that \[5\]

$$\forall B \langle \forall y \exists z (z < y \land h(y) < h(z)).$$

Let us verify that the set $A = \{2z : z \in B\} \cup \{2z + 1 : z \in B\}$ is a wsr*-set.

Let $x, y$ be two even numbers, $x = 2z_1, y = 2y_1, z_1 < y_1$. If there exists a number $z, z_1 < z \leq y_1$, such that $h(z) \leq h(z_1)$, then $z_1 \in B$. In this case we put $f(x, y) = y$. Otherwise, we must put $f(x, y) = x$, because it follows from (3) that if $x \in B$, then all the more $y \in B$. Also, for $y_1 \leq z_1$, we set $f(x, y) = f(y, x)$.

If in the case $x = 2z_1 + 1, y = 2y_1 + 1$ there is a number $z, z_1 < z \leq y_1$, such that $h(z) \leq h(z_1)$, then we put $f(x, y) = x$, and otherwise, $f(x, y) = y$. As before, we assume that $f(x, y) = f(y, x)$.

Finally, let $x = 2z_1, y = 2y_1 + 1$. We enumerate $B$. If $x_1$ is computed in $B$, we put $f(x, y) = f(y, x) = y$, if not, we set $f(x, y) = y$. It is readily verified that $A$ turns out to be a wsr*-set by the partially recursive function $f$.

It is easy to show that $A$ is not a wsr-set \[2\]. Indeed, if $A$ and $f$ satisfy condition (2), then we have $f(2x, 2x + 1) \downarrow$ for all $x$, and $x \in B \iff f(2x, 2x + 1) = 2x$, in contradiction to the fact that $B$ is nonrecursive. Now suppose that for $A$ there exists a general recursive function $f$ satisfying condition (1), and

$$R = \{(x_1, y_1) : y_1 < x \land (\forall z)(y_1 < z \leq x_1 \implies h(y_1) < h(z))\}, \quad x = 2x_1, \quad y = 2y_1 + 1.$$

Let $(x_1, y_1) \in R$ and $f(x, y) = x$. Then

$$(x_1 \in B \implies x \in A \implies y \in A \implies y_1 \notin B) \land (x_1 \notin B \implies y_1 \notin B).$$

Thus, $y_1 \notin B$ in both cases. But $\mathbb{N} \setminus B$ is an immune set, and the set

$$\{y_1 : (\exists x_1)(\langle x_1, y_1 \rangle \in R \land f(x, y) = x)\}$$

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