On the Fundamental Homology Classes of a Real Algebraic Variety

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ABSTRACT. It is proved that there is only one relation between the homology classes determined by the real points of a special real algebraic variety. This relation is equal to the sum of all the homology classes.

KEY WORDS: real algebraic variety, equivariant cohomology, cycle map, Chow ring.

Introduction

In what follows, $X$ is a smooth real projective variety. Let $X_1, \ldots, X_s$ be the connected components of the real locus $X(\mathbb{R})$. They define the homology classes $[X_1], \ldots, [X_s] \in \mathbb{H}_n(X(\mathbb{C}), \mathbb{F}_2)$, where $n = \dim X$.

The following question arises: What relations hold for these homology classes? In the case of a curve, there can be only one relation, namely $[X_1] + \cdots + [X_s] = 0$, i.e., $[X(\mathbb{R})] = 0$. In [1] it is shown that in the case of a surface with $H^1(X(\mathbb{C}), \mathbb{F}_2) = 0$ there also can be only one relation $[X(\mathbb{R})] = 0$. With the help of equivariant homology, a similar assertion was proved in [2] for complete intersections of arbitrary dimension. In the present note we generalize the above assertion to the class of “special” varieties. We say that a variety $X$ is special if

$$\text{cl}_C : A^{< n/2}(X) \rightarrow H^{< n}(X(\mathbb{C}), \mathbb{F}_2)$$

is epimorphic, where $A^{< n/2}(X)$ is the direct sum of the Chow groups $A^q(X)$ over $q < n/2$, and $H^{< n}(X(\mathbb{C}), \mathbb{F}_2)$ is the direct sum of the cohomology groups $H^p(X(\mathbb{C}), \mathbb{F}_2)$ over $p < n$.

Theorem. If $X$ is a special variety, then the homology classes $[X_1], \ldots, [X_s] \in \mathbb{H}_n(X(\mathbb{C}), \mathbb{F}_2)$ can satisfy only one relation $[X(\mathbb{R})] = 0$.

Note that this theorem implies the above assertions concerning a curve, a surface, and a complete intersection. In this note we prove the theorem with the help of equivariant cohomology and cycle maps. It was the application of cycle maps that allowed to generalize the result in [2].

§1. Equivariant cohomology and cycle maps

Consider the equivariant cohomology ring

$$H^*(X(\mathbb{C}) ; G, \mathbb{F}_2) = \bigoplus_{q \geq 0} H^q(X(\mathbb{C}) ; G, \mathbb{F}_2),$$

where $G = G(\mathbb{C}/\mathbb{R})$ is the Galois group. It is an algebra over the polynomial ring

$$H^*(pt ; G, \mathbb{F}_2) = H^*(G, \mathbb{F}_2) = \mathbb{F}_2[t],$$

where $pt$ is a singleton and $t$ is the generator of the group $H^1(G, \mathbb{F}_2)$. Note that the structure homomorphism $H^*(pt ; G, \mathbb{F}_2) \rightarrow H^*(X(\mathbb{C}) ; G, \mathbb{F}_2)$ is induced by the map $X(\mathbb{C}) \rightarrow pt$. Consider the following three homomorphisms:

$$\alpha^* : H^*(X(\mathbb{C}) ; G, \mathbb{F}_2) \rightarrow H^*(X(\mathbb{C}), \mathbb{F}_2),$$

$$\beta^* : H^*(X(\mathbb{C}) ; G, \mathbb{F}_2) \rightarrow H^*(X(\mathbb{R}) ; G, \mathbb{F}_2),$$

$$\text{cl}^* : A^*(X) \rightarrow H^*(X(\mathbb{C}) ; G, \mathbb{F}_2),$$


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where \( \alpha^* \) is the homomorphism of forgetting the real structure, \( \beta^* \) is the restriction homomorphism, \( A^*(X) \) is the Chow ring, and \( \text{cl}^* \) is the cycle map. The homomorphisms \( \alpha^* \), \( \text{cl}^* \) are ring homomorphisms, and \( \beta^* \) is a homomorphism of \( F_2[t] \)-algebras. Note that

\[
H^q(X(R); G, F_2) = \bigoplus_{k=0}^q H^k(X(R), F_2),
\]

and the homomorphism of multiplication by \( t \)

\[
v_t: H^q(X(R); G, F_2) \to H^{q+1}(X(R); G, F_2)
\]
is an inclusion of the direct sum into the direct sum, i.e., \( t \cup a = a \), where \( a \in H^k(X(R), F_2) \). Let \( n = \dim X \). By \( K^n(X(C); G, F_2) \) denote the kernel of the homomorphism

\[
\alpha^n: H^n(X(C); G, F_2) \to H^n(X(C), F_2).
\]

Then we have

\[
K^n(X(C); G, F_2) = t \cup H^{n-1}(X(C); G, F_2).
\]

This relation follows from the spectral sequence

\[
\Pi^p,q = H^p(G, H^q(X(C), F_2)) \implies H^{p+q}(X(C); G, F_2),
\]

since the homomorphism \( \alpha^n \) is its boundary homomorphism. Let \( 2q < n \). Then \( t^{n-2q} \cup \text{cl}^q(A^q(X)) \) is a subgroup of \( K^n(X(C); G, F_2) \). This fact follows from Eq. (1). By \( K^n_{\text{alg}}(X(C); G, F_2) \) denote the sum of the subgroups \( t^{n-2q} \cup \text{cl}^q(A^q(X)) \) for \( 2q < n \). If \( X \) is a special variety, then we have

\[
K^n_{\text{alg}}(X(C); G, F_2) = K^n(X(C); G, F_2).
\]

This relation follows from the definition of a special variety and the spectral sequence (2). In [3] the composition of the following homomorphism was calculated:

\[
\beta^{2q} \circ \text{cl}^q: A^q(X) \to H^{2q}(X(R); G, F_2) = \bigoplus_{k=0}^{2q} H^k(X(R), F_2),
\]

namely, if \( \text{cl}^q_k: A^q(X) \to H^q(X(R), F_2) \) is the real cycle map, then

\[
\beta^{2q} \circ \text{cl}^q = \text{Sq} \circ \text{cl}^q_k.
\]

where \( \text{Sq} \) is the total Steenrod square. By

\[
\beta^{n,0}: H^n(X(C); G, F_2) \to H^0(X(R), F_2)
\]
denote the composition of the homomorphism \( \beta^n \) with the projection

\[
\bigoplus_{k=0}^n H^k(X(R), F_2) \to H^0(X(R), F_2).
\]

Then (4) implies the relation

\[
\beta^{n,0}(K^n_{\text{alg}}(X(C); G, F_2)) = F_2 \cdot z^0,
\]

where \( z^0 \in H^0(X(R), F_2) \) is the cohomology class that takes value 1 on each one-point cycle. For a special variety, from (3), (5) we obtain

\[
\beta^{n,0}(K^n(X(C); G, F_2)) = F_2 \cdot z^0.
\]