Some Reducibilities and Splittings of Recursively Enumerable Sets

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ABSTRACT. The existence of a recursively enumerable (RE) $T$-degree $a$ that does not contain an RE semirecursive set $A \in a$ with the $Q$-universal splitting property is proved. Each nonrecursive RE contiguous degree contains an RE set $A$ with the universal $T-Q$-reduction property, but $A$ is not $T-Q$-maximal. Each nonrecursive RE $W$-degree contains an RE set $A$ with the universal $W-sQ$-reduction property, but $A$ is not $W-sQ$-maximal. Each creative set is partially semimaximal.

KEY WORDS: recursively enumerable degrees, universal splitting property, reducibilities, creative sets, semimaximal sets.

A recursively enumerable (RE) splitting of an RE set $A$ is a pair of RE sets $B$ and $C$ such that $A = B \cup C$ and $B \cap C = \emptyset$. If $B$ and $C$ form a splitting of a set $A$, we write $A = B \cup C$. Suppose that $A = B \cup C$ and $B$, $C$ are nonrecursive RE sets. Then the pair $B$, $C$ is called a nontrivial splitting of the set $A$. If $A = B \cup C$, then $\deg_T(A) = \deg_T(B) \cup \deg_T(C)$. Therefore, an RE splitting is a notion very helpful in the investigation of the structure of RE degrees.

An RE set $A$ has the universal splitting property (USP)\cite{1,2} if for any RE set $D \leq_T A$, there exists a splitting $A = B \cup C$ of the set $A$ such that $B \equiv_T D$. Lerman and Remmel\cite{1,2} showed that below each nonrecursive RE degree, there exist degrees all of whose RE sets do not have USP, as well as those that contain sets with USP. In addition, if $A$, $B$ are RE sets and $A <_T B$, then there exists an RE set $C$ such that $A <_T C <_T B$ and $C$ does not have USP. Downey\cite{3} generalized the last result and showed that each nonrecursive RE $T$-degree contains an RE set without USP.

An RE set $A$ has the strong universal splitting property\cite{4}, if for all RE sets $B_0, B_1$ satisfying the relation $A = B_0 \oplus B_1$, there exists a splitting $A = A_0 \cup A_1$ of the set $A$ such that $A_0 \equiv_T B_0$ and $A_1 \equiv_T B_1$. Ambos-Spies and Fejer\cite{4} studied algebraic properties of $T$-degrees of RE sets with the (strong) universal splitting property and obtained a number of interesting results.

In\cite{5}, interesting properties of semimaximal and partially semimaximal sets were introduced. Downey and Stob\cite{5} proved that if $A$ and $B$ are semimaximal sets, then there exists a $\Phi \in \text{Aut}(\mathcal{E})$ such that $\Phi(A) = B$, and if $A$ is a partially semimaximal set, then there exist $B \equiv_T \emptyset$ and $\Phi \in \text{Aut}(\mathcal{E})$ such that $\Phi(A) = B$, where $\text{Aut}(\mathcal{E})$ is the automorphism group of the lattice $\mathcal{E}$ of RE sets.

In this paper we prove that there exists an RE $T$-degree $a$ that contains no RE semirecursive set $A \in a$ having the $Q$-universal splitting property. In each nonrecursive RE contiguous degree there is an RE set $A$ with the universal $T-Q$-reduction property, but $A$ is not $T-Q$-maximal. In each nonzero RE $W$-degree there is an RE set $A$ with the universal $W-sQ$-reduction property, but $A$ is not $W-sQ$-maximal. The splitting of creative sets is studied and each creative set is shown to be a partially semimaximal set, which answers a question from\cite{5}. All the sets throughout the paper are RE sets.

All the notions and notation that we use without definition can be found in\cite{6,7}.

Let $r$ be a reducibility.

Definition. An RE set $A$ has the $r$-universal splitting property ($r$-USP), if for each RE set $B \leq_r A$ there exists an RE splitting $A = A_0 \cup A_1$ of $A$ such that $A_0 \equiv_r B$.

Suppose that $\varphi^B_0, \varphi^B_1, \ldots$ is a numeration of functions recursive in $B$, the indicator of the set $A$ is denoted by $C_A$, and $\{D_\omega\}$ is the canonical numbering of the collection of all finite subsets of the set $\omega = \{0, 1, 2, \ldots\}$.


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We say that \( A \leq_W B \) if there exists a \( z \) such that \( C_A(z) = \varphi_z^B(x) \) and all the numbers for which their membership or nonmembership in \( B \) was used in the computation of \( \varphi_z^B(x) \) lie in \( D_{f(x)} \), where \( f \) is a general recursive function (GRF).

We say that \( A \leq_Q B \), if there exists a GRF \( f \) such that

\[
(\forall z)(x \in A \iff W_{f(x)} \subseteq B).
\]

If, in addition, there exists a GRF \( g \) such that

\[
(\forall z)(\forall y)(y \in W_{f(x)} \implies y < g(x)),
\]

then \( A \leq_{bQ} B \).

If \( A \leq_{bQ} B \) and there exists a fixed number \( n \in \omega \) such that \( (\forall z)(|W_{f(x)}| < n) \), then \( A \leq_{bsQ} B \).

In [1] and [4], the notions of universal \( W \)-reduction property and \( W \)-maximal set were introduced. Similar notions can naturally be defined for other reducibilities as well.

An \( R \)-reducibility is said to be weaker than an \( r \)-reducibility (and \( r \) to be stronger than \( R \)) if we have \( A \leq_R B \implies A \leq_R B \) for all \( A, B \subseteq \omega \).

Let an \( R \)-reducibility be weaker than an \( r \)-reducibility.

**Definition.** An RE set \( A \) is called \( R \)-\( r \)-maximal if

\[
(\forall B \text{ RE})(B \leq_R A \implies A \equiv_r B \& A \equiv_r A).
\]

**Definition.** An RE set \( A \) has the universal \( R \)-\( r \)-reduction property if

\[
(\forall B \text{ RE})(\exists A_0 \text{ RE})(B \leq_R A \implies A_0 \equiv_r B \& A_0 \equiv_r A).
\]

It follows from Theorem 3.5 and Corollary 2.10 in [8] that if an RE \( T \)-degree \( c \) satisfies \( c' = 0' \), then there exist an RE \( T \)-degree \( a \geq c \) and an RE set \( A \in a \) such that for each RE set \( B \leq_T A \) with infinite complement there is a splitting \( A = A_0 \cup A_1 \) such that \( A_0 \equiv_m B \). Thus there exists an RE set with the \( r \)-universal splitting property, where \( r \) is an arbitrary reducibility intermediate between the \( m \)-and \( T \)-reducibilities.

**Proposition 1** (see [9]). Let \( A = B \cup C \). Then \( B \leq_{bsQ} A \) and \( C \leq_{bsQ} A \).

**Corollary.** (1) If an RE set \( A \) has the \( T \)-USP, then \( A \) has the universal \( T \)-\( Q \)-(\( T \)-\( sQ \)-, \( T \)-\( bsQ \)-) reduction property.

(2) If an RE set \( A \) has the \( Q \)-USP, then \( A \) has the universal \( Q \)-\( sQ \)-reduction property.

(3) If an RE set \( A \) has the \( W \)-USP, then \( A \) has the universal \( W \)-\( sQ \)-reduction property.

A set \( A \) is called semirecursive [10], if there exists a two-place GRF \( g \) such that

\[
(\forall z)(\forall y)(g(x, y) \in \{x, y\} \& (\{x, y\} \cap A \neq \emptyset \implies g(x, y) \in A)).
\]

**Lemma 1** (see [11]). Suppose that \( B \) is a semirecursive RE set, \( B \neq \omega \), \( B \neq \emptyset \), \( A \) is an RE set, and \( A \) is Turing reducible to \( B \). Then \( A \) is \( Q \)-reducible to \( B \).

**Lemma 2** (see [12]). Suppose that \( B \) is a semirecursive RE set, \( B \neq \omega \), \( B \neq \emptyset \), \( A \) is an RE set, and \( A \) is \( W \)-reducible to \( B \). Then \( A \) is \( sQ \)-reducible to \( B \).

An RE \( T \)-degree is said to be contiguous if all RE sets in it are \( W \)-equivalent.

**Remark.** There exists an RE nonrecursive set which is simultaneously \( T \)-\( Q \)-maximal, \( T \)-\( sQ \)-maximal, \( Q \)-\( sQ \)-maximal, and \( W \)-\( sQ \)-maximal.

Indeed, suppose that \( a \) is a contiguous RE \( T \)-degree, \( A \) is an RE semirecursive set, and \( A \in a \) [10]. Then \( A \) is the desired set by Lemma 2.