Coincidence of Classes of Functions Defined by the Generalized Shift Operator or by the Order of Best Polynomial Approximation

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ABSTRACT. In this paper the coincidence of two classes of functions is proved. One of them is determined by the power order of best polynomial approximation. To define the other class, first a new nonsymmetric generalized shift operator is introduced, and next, with its help we introduce a generalized modulus of continuity whose power order determines the second class of functions.

KEY WORDS: generalized shift operator, best approximation by algebraic polynomials, 2π-periodic function, Fourier–Jacobi series, Hölder’s inequality, Minkowski’s inequality.

At the beginning of this century it was found that the following conditions are equivalent for 2π-periodic functions $F$:

$$E_n(F)_{p^*} = O(n^{-r}) \quad \text{and} \quad \omega(F, \delta)_{p^*} = O(\delta^s), \quad (1)$$

where $0 < r < 1$ and $E_n(F)_{p^*}$ is the best approximation in the $L_{p^*}$ metric of the function $F$ by trigonometric polynomials of order at most $n - 1$ and $\omega(F, \delta)_{p^*}$ is the ordinary modulus of continuity of $F$ in the $L_{p^*}$ metric.

In the study of nonperiodic functions and their polynomial approximations, it was found that there is no similar equivalence if we consider the ordinary modulus of continuity. However, an analog of the assertion (1) is valid if the ordinary modulus of continuity is replaced by a generalized one. Such generalized moduli of continuity can be defined in a variety of ways. Let us cite one of them, related to the following analogy with the 2π-periodic case.

To each 2π-periodic function $F$ integrable on the closed interval $[0, 2\pi]$ we can assign at each point $t$ its Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

in the trigonometric system $\{e^{ikt}\}_{k=-\infty}^{\infty}$. Consider the series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt} e^{ikh}.$$ 

It is readily seen that this is the Fourier series of the function $F$ at the point of shift $t + h$.

To each nonperiodic function $f$ integrable with weight $(1 - z)_{\alpha}(1 + z)_{\beta}$ on the closed interval $[-1, 1]$ we can assign at each point $z \in [-1, 1]$ its Fourier–Jacobi series

$$\sum_{k=0}^{\infty} a_k P_k^{(\alpha, \beta)}(z)$$

in the system of Jacobi polynomials $\{P_k^{(\alpha, \beta)}(z)\}_{k=0}^{\infty}$, i.e., in the system of polynomials orthogonal to one another with weight $(1 - z)_{\alpha}(1 + z)_{\beta}$ on the closed interval $[-1, 1]$.

Consider the series

$$\sum_{k=0}^{\infty} a_k \varphi_k(h) P_k^{(\alpha, \beta)}(z), \quad (2)$$


where \( \{ \varphi_k(h) \}_{k=0}^{\infty} \) is a system of functions.

If it turns out that the series (2) is the Fourier–Jacobi series of some function, then by analogy with the \( 2\pi \)-periodic case, we can assume that this is the Fourier–Jacobi series of the function \( f \) at the point "of generalized shift," i.e., at the point \( z \pm h \) (the bar denotes this "generalized shift").

In some cases these functions can be explicitly written out; they are called generalized shift operators.

Only generalized shift operators for the case in which the Fourier–Jacobi series (2) is "symmetric," i.e., for \( \varphi_k(h) = P_k^{(\alpha, \beta)}(h) \), were studied earlier (see, for example, [1–4]). The study of "nonsymmetric" cases for which the function \( \varphi_k(h) \) does not coincide with the polynomial \( P_k^{(\alpha, \beta)}(h) \) is of interest.

In this paper we consider a generalized shift operator whose Fourier–Jacobi series (2) is "nonsymmetric" and prove an analog of assertion (1) with its help.

Let \( L_p \) be the set of functions \( f \) on the closed interval \([-1, 1]\) such that for \( 1 < p < \infty \) each function \( f \) is measurable on the closed interval \([-1, 1]\),

\[
\|f\|_p = \left( \int_{-1}^{1} |f(x)|^p \, dx \right)^{1/p} < \infty,
\]

and for \( p = \infty \) each function \( f \) is continuous on \([-1, 1]\) and

\[
\|f\|_\infty = \max_{-1 \leq x \leq 1} |f(x)|.
\]

Let \( L_{p, \alpha} \) be the set of functions \( f \) such that \( f(x)(1 - x^2)^\alpha \in L_p \); moreover,

\[
\|f\|_{p, \alpha} = \|f(x)(1 - x^2)^\alpha\|_p.
\]

Let \( E_n(f)_{p, \alpha} \) denote the best approximation of a function \( f \in L_{p, \alpha} \) by algebraic polynomials \( P_n \) of degree at most \( n - 1 \), i.e.,

\[
E_n(f)_{p, \alpha} = \inf_{P_n} \|f - P_n\|_{p, \alpha}.
\]

Let \( E(p, \alpha, r) \) denote the class of functions \( f \in L_{p, \alpha} \) satisfying the condition

\[
E_n(f)_{p, \alpha} \leq Cn^{-r},
\]

where \( r > 0 \) and \( C \) is some constant independent of \( n \) (\( n \in \mathbb{N} \)).

For a function \( f \in L_{p, \alpha} \), we introduce the generalized shift operator

\[
\hat{T}_t(f, x) = \frac{1}{\pi(1 - x^2)} \int_0^\pi f(x \cos t + \cos \varphi \sin t \sqrt{1 - x^2}) \left( 1 - (x \cos t + \cos \varphi \sin t \sqrt{1 - x^2})^2 - 2 \sin^2 t \sin^2 \varphi + 4(1 - x^2) \sin^2 t \sin^4 \varphi \right) \, d\varphi
\]

and, using this operator, we define the generalized modulus of continuity

\[
\overline{\omega}(f, \delta)_{p, \alpha} = \sup_{|t| \leq \delta} \|f(x) - \hat{T}_t(f, x)\|_{p, \alpha}.
\]

Let \( H(p, \alpha, r) \) denote the class of functions \( f \in L_{p, \alpha} \) satisfying the condition

\[
\overline{\omega}(f, \delta)_{p, \alpha} \leq C\delta^r,
\]

where \( r > 0 \) and \( C \) is some constant, independent of \( \delta \). The goal of this paper is to prove the following assertion.