On the Bruck–Slaby Theorem for Commutative Moufang Loops

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ABSTRACT. With the help of the relationship between commutative Moufang loops and alternative commutative algebras, we prove, rather easily, the following weakened version of the Bruck–Slaby theorem: a finitely generated commutative Moufang loop is centrally nilpotent.

KEY WORDS: commutative Moufang loops, alternative algebras, central nilpotence class.

The following theorem is one of the profound results in the theory of commutative Moufang loops (abbreviated CMLs):

**Bruck–Slaby Theorem** [1, Chap. VIII]. A commutative Moufang loop with \( n \) \((n \geq 2)\) generators is centrally nilpotent of class at most \( n - 1 \).

This theorem is a key instrument in the investigation of such loops. The proof of this assertion is very cumbersome; it is based on a complicated inductive process and uses several hundred nonassociative identities. In [2, Chap. 1] Yu. I. Manin used group methods to prove a weaker assertion, namely that any finite CML of period 3 is centrally nilpotent. Although less calculative, his proof is by no means simple; it uses deep facts from finite group theory, such as the Feit–Thompson theorem on the solvability of any group of odd order and the Brauer–Suzulld theorem stating that all groups containing generalized quaternion groups are not simple. Quite recently Smith [3], also using the apparatus of finite groups, obtained a simpler proof of Manin’s result.

Note that the supremum of the central nilpotence class of a CML with \( n \) generators is equal to \( n - 1 \) (see [4–6]).

In the present paper the relationship between CML and alternative commutative algebras is used to prove (rather simply) that any finitely generated CML is centrally nilpotent (Theorem 1). In the proof we only use the fact that any alternative commutative nil-algebra of index 3 is locally nilpotent.

Recall some notions and results from [1].

Let \((Q, \cdot)\) \(\equiv Q\) be an arbitrary loop. A subloop \(H\) of the loop \(Q\) is said to be normal in \(Q\) if for any elements \(x, y\) in \(Q\) we have \(x \cdot y H = H \cdot x, H x \cdot y = H \cdot x y,\) and \(x H = H x\). The associator \([a, b, c]\) of the elements \(a, b, c\) of \(Q\) is defined by the relation \(a b \cdot c = (a \cdot b) c[a, b, c]\).

The loop \(Q\) is called a *commutative Moufang loop* (CML) if the commutativity identity \(x y z = y x z\) and one of the equivalent *Moufang identities*

\[
x y \cdot z z = (x y \cdot z) z, \quad (z x \cdot y) = z (x \cdot y z), \quad y z \cdot x y = y (x y \cdot z) z
\]

hold in it. A CML can also be characterized by one identity, namely,

\[
x^2 \cdot y z = x y \cdot z z.
\]

A CML is an *IP-loop*, i.e., satisfies the identity \(z^{-1} \cdot x y = y \cdot z^{-1} = 1\), and is *diassociative*, i.e., any two of its elements generate an associative subloop.

If \(A, B, C\) are nonempty subsets of a CML \(Q\), then by \([A, B, C]\) denote the subloop of \(Q\) generated by all associators \([a, b, c]\), where \(a \in A, b \in B, c \in C\). Define inductively \(Q_0 = Q, Q_n = [Q_{n-1}, Q, Q]\).

If there exists a natural number \(n\) such that \(Q_n = 1\), then the CML \(Q\) is said to be centrally nilpotent. The minimal number \(n\) with this property is called the *central nilpotence class*.

The *center* \(Z(Q)\) of a CML \(Q\) is the set \(Z(Q) = \{z \in Q \mid [z, y, z] = 1 \ \forall y, z \in Q\}\). It is readily seen that \(Z(Q)\) is a normal subloop of \(Q\).
For completeness, we present here an elementary proof of a known fact from [1].

Lemma 1. The quotient loop \( Q/Z(Q) \) of an arbitrary commutative Moufang loop \( Q \) by its center \( Z(Q) \) satisfies the identity

\[ x^3 = 1. \]  

Proof. Let us use the fact that a CML is diassociative and is an IP-loop. By (2), the identity \( z^2(x \cdot zy) = (xz)(z \cdot zy) \) holds in the CML \( Q \). Now substitute \( z \cdot xy \) for \( y \) and \( x^{-1}z^{-1} \) for \( x \). By (2) we have \( z \cdot xy = z^{-1}(xz \cdot zy) \). Then

\[
\begin{align*}
&z^2((x^{-1}z^{-1})((x^{-1}(xz \cdot zy)))) = (z \cdot z^{-1}x^{-1})(z(xz \cdot zy)), \\
&z^2((xz^{-1})(xz \cdot zy)) = x^{-1}(z^3 \cdot xy), \\
&z^2 \cdot zy = z^{-1}(z^3 \cdot zy), \\
&x^3y = x^{-1}(z^3 \cdot zy), \\
&z^3y \cdot x = z^3 \cdot yz,
\end{align*}
\]

i.e., \( x^3 \in Z(Q) \). The lemma is proved. \( \square \)

Let \( \Phi \) be an associative commutative ring with unity, and let \( Q \) be a loop. By analogy with group algebras [7], consider the loop algebra \( \Phi Q \). It is a free \( \Phi \)-module with basis \( \{ q | q \in Q \} \), i.e., the set of finite formal sums \( \sum_{q \in Q} \lambda_q (\lambda_q \in \Phi, q \in Q) \) with operations

\[
\lambda_q q + \sum_{q \in Q} \mu_q q = \left( \sum_{q \in Q} \lambda_q q \right) \left( \sum_{h \in Q} \mu_h h \right) = \sum_{k \in Q} \sum_{z, y \in Q, zy = k} \lambda_q \mu_y k.
\]

Let \( H \) be a normal subloop of \( Q \). By \( \omega H \) denote the ideal of \( \Phi Q \) generated by the elements \( 1 - h \) (\( h \in H \)). If \( H = Q \), then, by analogy with [7], call \( \omega Q \) the fundamental ideal of the algebra \( \Phi Q \).

Lemma 2. Let \( H \) be a normal subloop of a loop \( Q \). Then

1) if the elements \( h_i \) generate the subloop \( H \), then the elements \( 1 - h_i \) generate the ideal \( \omega H \);
2) the fundamental ideal is generated as a \( \Phi \)-module by the elements of the form \( 1 - q \) (\( q \in Q \)).

Proof. Suppose that the elements \( \{ h_i \} \) generate the subloop \( H \) and that \( I \) is the ideal generated by the elements \( \{ 1 - h_i \} \). It is clear that \( I \subseteq \omega H \). Conversely, let \( g \in H \) and \( g = g_1 g_2 \), where \( g_1, g_2 \) are words over \( h_i \). Suppose that \( 1 - g_1, 1 - g_2 \in I \). Then we have \( 1 - g = (1 - g_1)g_2 + 1 - g_2 \in I \), i.e., assertion 1) is proved.

Let us prove assertion 2). Since we have \((1 - q)q' = (1 - qq') - (1 - q')\), the fundamental ideal \( \omega Q \) is generated as a \( \Phi \)-module by the elements of the form \( 1 - q \), where \( q \in Q \). This completes the proof of the lemma. \( \square \)

In what follows, we assume that \( \Phi \) is a field of characteristic 3 and the loop \( Q \) is commutative. By definition, the elements of \( Q \) are linearly independent in the loop algebra \( \Phi Q \), hence if the loop \( Q \) is not associative, then the relation

\[ [a, b, c] + [a, c, b] + 1 = 0 \quad \forall a, b, c \in Q \]  

(4)
does not hold in \( \Phi Q \). However, in what follows we shall consider the modified algebra \( \Phi Q \). Namely, we shall require that (4) holds in \( \Phi Q \). It means that we shall in fact consider the quotient algebra of \( \Phi Q \) by the ideal generated by all elements of the left-hand side of (4).

It is clear that condition (4) does not place any constraints on the loop operation \( \cdot \) if for any elements \( a, b, c, u, v, w \) in \( Q \) the following implications hold:

\[
\begin{align*}
[a, b, c] = [u, v, w] & \implies [a, c, b] = [u, w, v], \\
[a, b, c] \neq 1 & \implies [a, c, b] \neq [a, c, b], \\
[a, b, c] = 1 & \implies [a, c, b] = 1.
\end{align*}
\]

(5)

Now, taking into account (4), for any elements \( a, b, c \in Q \) we have

\[
ab \cdot c + bc \cdot a + ca \cdot b = (a \cdot bc)[a, b, c] + a \cdot bc + (a \cdot cb)[a, c, b] = (a \cdot bc)((a, b, c) + 1 + [a, c, b]) = 0,
\]