Rest Points of Generalized Dynamical Systems

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Abstract. In this paper we consider generalized dynamical systems whose integral vortex (that is, the set of all trajectories of the system starting at a given point) is an acyclic set in the corresponding space of curves. For such systems we apply the theory of fixed points for multi-valued maps in order to prove the existence of rest points. In this way we obtain new existence theorems for rest points of generalized dynamical systems.

Key Words: dynamical system, rest points, acyclic images, Lefschetz theory

Ordinary differential equations without uniqueness and differential inclusions gave rise to the theory of generalized dynamical systems. The foundations of this theory were laid in the works by A. D. Myshkis, E. A. Barbashin, B. M. Budak, E. Roksin, and others.

The problem of investigating rest points for generalized dynamical systems was stated in Myshkis’s paper [1]. In that paper the problem was reduced to the study of fixed points of multi-valued maps whose images had a highly sophisticated topological structure. It turned out to be impossible to describe the structure of images of multi-valued maps appearing in the rest point problem in dimension greater than 1. Hence the problem turned out to be very difficult. However some existence theorems for rest points of generalized dynamical systems were obtained in [2, 3].

In this paper we consider a special class of generalized dynamical systems. The corresponding multi-valued map for systems in this class is the composition of a single-valued continuous map and a multi-valued map whose images are acyclic. Fixed point theory is well developed for such maps (see, for example, [2, 4]); hence we can prove some existence theorems for rest points of dynamical systems in this class.

Facts from the theory of generalized dynamical systems that we use in what follows may be found in the monographs [5–7].

§1. Generalized dynamical systems: basic facts.

Let Y be a subset of a Banach space E. Denote by K(Y) the set of all nonempty compact subsets of Y, by Kv(Y) the set of all nonempty compact convex subsets of Y.

A multi-valued map (or an M-map) of a metric space X into a metric space Z is a correspondence that assigns to each point \( z \in X \) a nonempty subset \( F(z) \subseteq Z \), called the image of \( z \). We shall write \( F: X \rightarrow K(Z) \) if all images of the M-map \( F \) are compact sets. If all images \( F(z) \) are convex compact sets, then we shall write \( F: X \rightarrow Kv(Z) \).

The graph of a multi-valued map \( F: X \rightarrow K(Z) \) is by definition the set

\[
\Gamma_X(F) = \{(z, z) \mid z \in F(x), \ x \in X\} \subseteq X \times Z.
\]

We have the natural projections \( t: \Gamma_X(F) \rightarrow X, \ t(z, z) = z \) and \( r: \Gamma_X(F) \rightarrow Z, \ r(z, z) = z \).

Clearly, \( F(x) = r \cdot t^{-1}(z) \) for any \( z \in X \).

In what follows, we denote M-maps by capital letters, single-valued maps by small letters.

Definition 1. A multi-valued map \( F: X \rightarrow C(Z) \) is called upper semicontinuous at a point \( x_0 \in X \) if for any open set \( V \subseteq Z \) such that \( V \supseteq F(x_0) \) there exists an open neighborhood \( U \) of the point \( x_0 \) such that \( F(U) \subseteq V \).

If a map \( F \) is upper semicontinuous at each point \( x \in X \), then it is called upper semicontinuous.
Definition 2. A multi-valued map $F: X \to C(Z)$ is called lower semicontinuous at a point $x_0 \in X$ if for any open set $V \subset Z$ such that $F(x_0) \cap V \neq \emptyset$ there exists an open neighborhood $U$ of the point $x_0$ such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.

If a map $F$ is lower semicontinuous at each point $x \in X$ then it is called lower semicontinuous.

Definition 3. A multi-valued map $F: X \to C(Z)$ is called continuous if it is both upper and lower semicontinuous.

Suppose $E$ is a Banach space, suppose $X \subset E$, and let $F: X \to C(E)$ be an M-map. A point $x_0$ is called a fixed point of the M-map $F$ if $x_0 \in F(x_0)$.

There exist several definitions of a generalized dynamical system (that is, a dynamical system without the uniqueness property). The definition given by Barbashin in [8] is the one most frequently used.

Let $X$ be a topological space; denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}_+$ the set of all nonnegative real numbers.

Definition 4. A multi-valued map $\Psi: X \times \mathbb{R} \to K(X)$ is called a generalized dynamical system if the following conditions are satisfied:

1) $\Psi(-0): X \to X$ is the map $\text{id}_X$;
2) $\Psi(\Psi(x, t_1), t_2) = \Psi(x, t_1 + t_2)$ for all $x \in X$ and $t_1, t_2 \in \mathbb{R}$ such that $t_1 \cdot t_2 \geq 0$;
3) for any $x \in X$, $t \in \mathbb{R}$, and $y \in \Psi(x, t)$ we have $x \in \Psi(y, -t)$;
4) the M-map $\Psi$ is upper semicontinuous;
5) for any $x_0 \in X$ the M-map $\Psi(x_0, \cdot): \mathbb{R} \to K(X)$ is continuous.

We obtain an example of a generalized dynamical system by considering the shift operator along the trajectories of an autonomous generalized differential equation $\dot{x} \in F(x)$ where $F: X \to \text{Kv}(\mathbb{R}^n)$ is an upper semicontinuous M-map and the solutions of the equation are nonlocally extendable to the whole real line. Another source of examples is obtained by considering ordinary differential equations without the uniqueness property.

Definition 5. A trajectory of a generalized dynamical system on an interval $[a, b] \subset \mathbb{R}$ is a continuous map $z: [a, b] \to X$ such that for any $t_0, t_1 \in [a, b]$ we have $z(t_1) \in \Psi(z(t_0), t_1 - t_0)$.

In what follows, we always assume $X$ to be a metric space.

We have the following properties of trajectories of generalized dynamical systems (for proofs see, for example, [5]).

Property 1. A trajectory of a generalized dynamical system is a continuous map.

Property 2. Let $x_1$ be a trajectory of a generalized dynamical system on the interval $[a, b]$, $x_2$ be a trajectory of the system on the interval $[b, c]$. Suppose $x_1(b) = x_2(b)$. Then the map $z: [a, c] \to X$,

$$
z(t) = \begin{cases} 
  x_1(t) & \text{if } t \in [a, b), \\
  x_2(t) & \text{if } t \in (b, c],
\end{cases}
$$

is a trajectory of the system on the interval $[a, c]$.

Property 3. Let $\Psi$ be a generalized dynamical system. Then for all real $a, b$, $a \leq b$, and all $x_0 \in X$, $x_1 \in \Psi(x_0, b - a)$ there exists a trajectory $z$ on the interval $[a, b]$ such that $z(a) = x_0$, $z(b) = x_1$.

Let $A \subset X$. Denote by $\Sigma(A, [a, b])$ the set of all trajectories $z$ of the system $\Psi$ on the interval $[a, b]$ satisfying $z(a) \in A$.

Property 4. Let $X$ be a metric space with the distance function $\rho$. Suppose $\Psi: X \times \mathbb{R} \to K(X)$ is a generalized dynamical system. Then for any compact subset $A \subset X$ and any interval $[a, b] \subset \mathbb{R}$ the set $\Sigma(A, [a, b])$ is compact in the uniform convergence topology.

If in Definition 4 we exclude condition 3) and replace the set of all real numbers $\mathbb{R}$ by the set of all nonnegative real numbers $\mathbb{R}_+$, then we obtain the definition of a one-sided generalized dynamical system. The notion of trajectory is defined for one-sided dynamical systems in the obvious way; properties analogous to properties 1–4 still hold for them; moreover, we have the following lemma.