Clarkson–Erdős Theorem for Many Variables

S. G. Merzlyakov

ABSTRACT. In the present paper, the Clarkson–Erdős theorem, which complements the well-known Müntz theorem, is extended to the case of many variables.

KEY WORDS: holomorphic extension, Müntz theorem, Clarkson–Erdős theorem, Fréchet space.

§1. Introduction

In [1], the following result was proved.

Clarkson–Erdős theorem. Suppose that for \( \lambda_n \in \mathbb{N} \),

\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty,
\]

and for the numbers \( a_{nk} \in \mathbb{C}, \ n = 1, \ldots, k, \ k \in \mathbb{N} \), the sequence

\[
\sum_{n=1}^{k} a_{nk} z^{\lambda_n}, \quad k \in \mathbb{N},
\]

is uniformly convergent on the interval \([0, 1]\). In this case, the limit function can be extended holomorphically to the unit circle with center at zero, and the above sequence is convergent to this function uniformly on the compact sets in this circle.

In the present paper we generalize the Clarkson–Erdős theorem to the case of many variables. We need two inequalities for polynomials in one variable.

Theorem A. For a number sequence \( \{\lambda_n : n \in \mathbb{N}\} \) satisfying condition (1) and for any number \( \varepsilon > 0 \), there is a number \( c > 0 \) such that

\[
\max_{|z| \leq 1+\varepsilon} |p(z)| \leq c \max_{1 \leq \varepsilon \leq \lambda_1+2\varepsilon} |p(z)|,
\]

where \( p \) is an arbitrary linear combination of the monomials \( \{z^{\lambda_n} : n \in \mathbb{N}\} \).

This inequality clearly follows from a result due to Schwartz for quasipolynomials with real exponents (see [2, p. 64]).

Theorem B. For an arbitrary polynomial \( p \) of degree \( n \) we have the relation

\[
|p(z)| \leq \max_{-1 \leq t \leq 1} |p(t)| \cdot |z + \sqrt{1 - z^2}|^n, \quad z \in \mathbb{C}.
\]

This is the well-known Bernshtein inequality (see [3]), which implies the following rougher inequality:

\[
|p(z)| \leq \max_{-1 \leq t \leq 1} |p(t)|(1 + 2|z|)^n, \quad z \in \mathbb{C}.
\]
§2. Series of homogeneous polynomials

In this section we consider series of homogeneous polynomials that converge on real domains.

Theorem 1. For a domain $U \subset \mathbb{R}^q$, there is a full circular domain $G \subset \mathbb{C}^q$, $U \subset G$, such that if the terms of the series

$$\sum_{n=0}^{\infty} A_n$$

are uniformly bounded on the compact sets of the domain $U$, where $A_n$ is a polynomial in $q$ variables of degree $n$, then this series is uniformly convergent on the compact sets of the domain $G$.

Proof. For an arbitrary point $z_0 \in U$, there exists numbers $r > 1$ and $\delta > 0$ such that the cube $K = \{z \in \mathbb{R}^q : |z_j - z_j^0| \leq \delta, \ j = 1, \ldots, q\}$ belongs to the domain $U$, where $z_1 = rz_0$. By assumption, there is a point $c > 0$ such that

$$|A_n(z)| \leq c, \quad z \in K, \quad n \in \mathbb{N}_0.$$

Applying the Bernstein inequality, we can readily derive the relation

$$|A_n(\delta z + z^0)| \leq c(1 + 2|z_1|)^n \cdots (1 + 2|z_q|)^n, \quad z \in \mathbb{C}^q. \tag{3}$$

Obviously, for a fixed number $\rho$, $1 < \rho < r$, there is a number $\delta_1 > 0$ such that relation (3) implies the inequality

$$|A_n(w + z^1)| \leq c\rho^n, \quad |w^j| \leq \delta_1, \quad j = 1, \ldots, q.$$

In this case, since the polynomials $A_n$ are homogeneous, it follows that

$$|A_n(r^{-1}w + z^0)| \leq c(r^{-1}\rho)^n, \quad |w^j| \leq \delta_1, \quad j = 1, \ldots, q,$$

and hence the series (2) is uniformly convergent in a complex neighborhood of the point $z_0$.

Let us now show that this series is uniformly convergent in a complex neighborhood of the origin.

It follows from relation (3) on the set $\{w \in \mathbb{C}^q : |w_j| \leq 1, \ j = 1, \ldots, q\}$ that

$$|A_n(w)| \leq c\rho_1^n, \quad n \in \mathbb{N}_0, \quad \text{where} \quad \rho_1 = (1 + 2\delta^{-1}(1 + |z_1^1|)) \cdots (1 + 2\delta^{-1}(1 + |z_q^1|)),$$

and, since the polynomials $A_n$ are homogeneous, it follows that the series (2) is uniformly convergent on the compact set $\{w \in \mathbb{C}^q : |w_j| \leq r_1, \ j = 1, \ldots, q\}$ for any number $r_1 < \rho_1^{-1}$.

It is clear that if the series (2) is uniformly convergent in a neighborhood of the point $z_0 \in \mathbb{C}^q$, then this series is also uniformly convergent in a neighborhood of the circle $\{\zeta z_0 : \zeta \in \mathbb{C}, \ |\zeta| \leq 1\}$. Thus the existence of the desired domain is established. \]

§3. Sequences of polynomials

In this section we generalize the Clarkson–Erdős result to the case of many variables.

To this end, we need the following definitions.

For a domain $U \subset \mathbb{R}^q$ (a domain $G \subset \mathbb{C}^q$), denote by $C(U)$ (by $H(G)$) the space of continuous complex-valued functions on the domain $U$ (the space of holomorphic functions on the domain $G$, respectively) endowed with the topology of uniform convergence on compact sets. It is clear that these are Fréchet spaces.

Theorem 2. Suppose $P_{nk}$ is a homogeneous polynomial in $q$ variables of degree $\lambda_n$, $k \in \mathbb{N}$, $n = 1, \ldots, k$, the number sequence $\{\lambda_n \in \mathbb{N} : n \in \mathbb{N}\}$ satisfies condition (1), and, for some function $f \in C(U)$, we have the relation

$$\lim_{k \to \infty} \sum_{n=1}^{k} P_{nk} = f \tag{4}$$

in the topology of the space $C(U)$. Then the function $f$ can be extended holomorphically to the domain $G$ constructed in Theorem 1, and the limit (4) with respect to the topology of the space $H(G)$ exists.