On a Certain Property of the Burkill $SCP$-Integral

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ABSTRACT. We establish a necessary condition for the $SCP$-integrability of a function and use it to construct an example of a function possessing a continuous $SCP$-majorant and $SCP$-minorant on a closed interval, but nonintegrable over this interval in the sense of the $SCP$-integral.

KEY WORDS: generalized integral of Perron type, Burkill $SCP$-integral, necessary condition for $SCP$-integrability, $SCP$-majorant, $SCP$-minorant, Perron integral.

The following property of the Perron integral is well known: a measurable function $f$ possessing at least one continuous $P$-minorant and at least one continuous $P$-majorant on the closed interval $[a, b]$ is necessarily $P$-integrable on $[a, b]$ (see [1, p. 365 of the Russian translation], as well as [2]). This result was later extended to some other generalized integrals of Perron type (see, for example, [3]). On the other hand, for the Perron binary integral the existence of a continuous majorant and minorant of a function does not guarantee its integrability (see [4]). Here we show that for the Burkill $SCP$-integral (see [5]), which allows one to consider any trigonometric series everywhere converging to a finite sum as the Fourier series of its sum, a similar example can also be constructed.

Let us present the necessary definitions and notation.

Suppose that $B \subset [a, b]$, $|B| = b - a$, $a, b \in B$. A $P$-integrable function $M$ is called an $SCP$-majorant with basis $B$ for $f$ on $[a, b]$ if

1) $M(a) = 0$;
2) $M$ is $SC$-continuous for all $z \in (a, b)$, i.e.,
\[
\lim_{h \to 0} \frac{1}{h} \left( (P) \int_{z}^{z+h} M \, dt - (P) \int_{z-h}^{z} M \, dt \right) = 0, \quad z \in (a, b);
\]
3) $M$ is $C$-continuous for all $z \in B$, i.e.,
\[
\lim_{h \to 0} \frac{1}{h} (P) \int_{z}^{z+h} M \, dt = M(z), \quad z \in B;
\]
4) $SCP M(z) \geq f(z)$ almost everywhere on $(a, b)$, where
\[
SCP M(z) = \lim_{h \to 0} \frac{1}{h^2} \left( (P) \int_{z}^{z+h} M \, dt - (P) \int_{z-h}^{z} M \, dt \right);
\]
5) $SCP M(z) > -\infty$ everywhere on $(a, b)$ except, maybe, for a countable set $E$.

An $SCP$-minorant with basis $B$ is defined similarly.

A function $f$ is called $SCP$-integrable on $[a, b]$ with basis $B$ if for its $SCP$-majorants $M$ and $SCP$-minorants $m$ with basis $B$ the following relation is valid:
\[
\sup m(b) = \inf M(b) = \Phi(b).
\]

Moreover, we assume that
\[
(SCP) \int_{a}^{b} f \, dx = \Phi(b).
\]
A function $f$ that is SCP-integrable with basis $B$ on $[a, b]$, is SCP-integrable with the basis $B \cap [a, x]$ on $[a, x]$ for $x \in B$. The function $\Phi(x)$, $x \in B$, is naturally called SCP-primitive for $f$ on $[a, b]$.

Since an SCP-integrable function is finite almost everywhere on $[a, b]$ and a variation of this function on the set of measure zero does not affect its SCP-integrability or its SCP-primitive function, we may assume that $f(x)$ is everywhere finite.

Repeating the proof of Lemma 2 from [6] for SCP-integrals, we find that we can assume, without loss of generality, that conditions 4) and 5) in the definitions of the majorant and the minorant are satisfied everywhere.

**Lemma.** Suppose that $\Phi$ is an SCP-primitive function for $f$ on the closed interval $[a, b]$. Then for any $\varepsilon > 0$ the interval $(a, b)$ can be expressed as the union of a countable number of closed (on $(a, b)$) sets $Q_k$ possessing at most two common points for different $k$ such that for $x \in Q_k$, $x \pm h \in [a, b]$ the following inequality holds:

$$
(P) \int_0^h (\Phi(x + t) - \Phi(x - t)) \, dt \leq \int_0^h (\sigma_k(x + t) - \sigma_k(x - t)) \, dt,
$$

where the $\sigma_k$ are nondecreasing continuous functions on $[a, b]$; moreover, if $s_k$ is the singular part of $\sigma_k$, then

$$
\sum_{k=1}^{\infty} (s_k(b) - s_k(a)) < \varepsilon.
$$

**Proof.** Repeating the proof of Lemma 2 from [7], in the case of an SCP-integral we find that for an arbitrary $\varepsilon > 0$ the interval $(a, b)$ can be expressed as the union of closed (on $(a, b)$) sets $P_k$, $k = 1, 2, \ldots, P_{k+1} \supset P_k$, so that for $x \in P_k$ inequality (1) holds with $\widehat{\sigma}_k(z) = kx + \sigma(z)$, where $\sigma$ is a nondecreasing continuous function on $[a, b]$ for which $0 \leq \sigma(b) - \sigma(a) < \varepsilon$. The difference of the sets $P_{k+1} \setminus P_k$ can be expressed as the union of at most a countable number of sets $Q_{kn} = u_n \cap (P_{k+1} \setminus P_k)$, where the $\{u_n\}$ are the intervals composing the set $(a, b) \setminus P_k$; moreover, $Q_{kn}$ differs from $Q_{kn}$ by at most two points. For different $k$ and $n$, the sets $Q_{kn}$ ($Q_{11} = P_1$), obviously, do not overlap. In the set $\{Q_{kn}\}_{k,n=1}^{\infty}$, let us introduce the single numbering $\{Q_{kn}\}_{k,n=1}^{\infty} = \{Q_k\}_{k=1}^{\infty}$. For $x \in Q_k$, inequality (1) is valid with a nondecreasing continuous function $\widehat{\sigma}_k(z) = c(k) x + \sigma(x)$.

Suppose that $\{(\alpha_j, \beta_j)\}_{j=1}^{\infty}$ is the set of intervals of the set $(a, b) \setminus Q_k$,

$$
\omega_j(\delta) = \sup\{[\overline{\sigma}_k(x) - \overline{\sigma}_k(y)] : x, y \in (\alpha_j, \beta_j), \ |x - y| < \delta \}
$$

is the modulus of continuity of the function $\overline{\sigma}_k$ on $(\alpha_j, \beta_j)$, $\overline{\omega}_j(\delta)$ is a convex (up) function satisfying the inequality $0 \leq \omega_j(\delta) \leq \overline{\omega}_j(\delta) \leq 2\omega_j(\delta)$ for $0 \leq \delta \leq \beta_j - \alpha_j$ (see [7, Stechkin’s lemma]). The function

$$
\overline{\omega}_j(z) = \begin{cases} 0 & \text{for } a \leq z < \alpha_j, \\ \overline{\omega}_j(z - \alpha_j) + \overline{\omega}_j(\beta_j - \alpha_j) - \overline{\omega}_j(\beta_j - z) & \text{for } \alpha_j \leq z < \beta_j, \\ 2\overline{\omega}_j(\beta_j - \alpha_j) & \text{for } \beta_j \leq z \leq b,
\end{cases}
$$

does not decay, is absolutely continuous, and we have $\overline{\omega}_j(\alpha_j) = 0$, $\overline{\omega}_j(b) \leq 4(\overline{\sigma}(\beta_j) - \overline{\sigma}(\alpha_j))$.

Let us define the increasing continuous function

$$
\sigma_k(z) = V\sigma([a, z] \cap Q_k) + c(k)[a, z] \cap Q_k] + \sum_{j=1}^{\infty} \overline{\omega}_j(z),
$$

where $V\Psi(E)$ is a function of the set $E \subset [a, b]$ with generating function $V^*\Psi$ ($\Psi$ is a continuous function of finite variation).

Next, we verify that the sequence of sets $\{Q_k\}_{k=1}^{\infty}$ and monotone functions $\{\sigma_k(z)\}_{k=1}^{\infty}$ satisfies the assumptions of the lemma.