DEFORMED SUPERSYMMETRY, \(q\)-OSCILLATOR ALGEBRA, AND RELATED SCATTERING PROBLEMS IN QUANTUM MECHANICS

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We describe extensions of the supersymmetric quantum mechanics (SSQM) (in one dimension) which are characterized by deformed algebras. The supercharges involving higher-order derivatives are introduced, leading to a deformed algebra which incorporates a higher-order polynomial of the Hamiltonian. When supplementing them with dilatations, one finds the class of \(q\)-deformed SUSY systems. For a special choice of \(q\)-self-similar potentials, the energy spectrum is (partially) generated by the \(q\)-oscillator algebra. In contrast to the standard harmonic oscillators, these systems exhibit a continuous spectrum. We investigate the scattering problem in the \(q\)-deformed SSQM and introduce the notion of self-similarity in the momentum space for scattering data. An explicit model for the scattering amplitude of a \(q\)-oscillator is constructed in terms of a hypergeometric function. This model corresponds to a reflectionless potential with infinitely many bound states. A general method of realization of the \(q\)-oscillator algebra on the space of wave functions for a one-dimensional Schrödinger Hamiltonian is developed. It shows the existence of non-Fock irreducible representations associated with the continuous part of the spectrum and directly related to the deformation. Bibliography: 24 titles.

1. INTRODUCTION

The deformed symmetries were introduced [1, 2] to characterize the integrability of some lattice models and conformal field theories [3, 4]. Later on, a number of examples of \(q\)-deformed symmetries or \(q\)-deformed spectrum-generating algebras were found in various quantum systems, in particular, in low-dimensional quantum-mechanical (QM) models.

A simple algebra which is realized on isospectral QM systems is the supersymmetric quantum-mechanical algebra [5, 6] of supercharges and a superhamiltonian. Two supersymmetric partners of a superhamiltonian are related (more technically intertwined) by supercharges which actually generate Darboux transformations [7] between two isospectral systems. Deformations of this algebra are provided by extended supercharges involving higher-order derivatives [8-10]. Such an extension leads to a higher-derivative SUSY (HSUSY) algebra [10] which incorporates a higher order polynomial of the Hamiltonian. On the other hand, when supplementing supercharges with dilatations, one reveals the class of \(q\)-deformed potential systems [11] on which the \(q\)-deformed oscillator algebra can be realized as a spectrum-generating algebra.

The oscillator or Heisenberg–Weyl algebra was deformed in several ways [12, 13], and its representations have formally been classified [13-15]. Because of the significance of the conventional oscillator in many areas of modern quantum physics, new realizations of the \(q\)-oscillator algebra on the wave function space of a particular dynamical model are useful for an understanding of the role of this algebra for physical systems.

It is our goal to describe the interplay between polynomial deformations of the SUSY algebra referred to pairs of isospectral systems and \(q\)-deformations introduced into a Schrödinger-type system by dilatations of coordinates. In this way, different realizations of the \(q\)-oscillator as a quantum system satisfying the Schrödinger equation are found.

As also for the harmonic oscillator \((q = 1)\), the connection between the values of different energy levels and the corresponding wave functions [11] (or reflection and transmission coefficients [10]) becomes a consequence of the \(q\)-oscillator algebra.

In order to reproduce a \(q\)-oscillator, it is necessary to identify the potentials in the Darboux-connected Hamiltonians up to a constant. This self-similarity property holds for the conventional harmonic oscillator and explains its equidistant energy spectrum. The dilated self-similarity condition [11, 16] naturally selects the potentials that yield the energy spectra and wave functions of a \(q\)-deformed oscillator. The relation
between scattering amplitudes of two q-superpartners leads to the “dual” condition of self-similarity [10] (in the momentum space).

In Sec. 2, we outline the basics of one-dimensional SSQM [5, 6]. In Sec. 3, a SSQM with supercharges of second-order in derivatives is constructed in the most general form, and cases are pointed out where the superpartners cannot be constructed by iterations of two ordinary Darboux transformations. Thereby we classify all irreducible transformations involved in the construction of HSUSY systems with polynomially deformed superalgebras.

In Sec. 4, we study the q-deformations of SUSY and HSUSY induced by the dilatation of the coordinates. The coexistence between a q-deformed SUSY algebra with the ordinary Hamiltonian and the ordinary SUSY algebra with the q-deformed Hamiltonian is established and further generalized to HSUSY.

In Sec. 5, we introduce the closure, or self-similarity, condition into the q-deformed SUSYQM and then construct a q-oscillator model with local potential. The proof of the existence of a regular potential is discussed. Its polynomial generalization is obtained by the q-deformation of a polynomial superalgebra.

In Sec. 6, we consider the q-oscillator algebra in the form [13] and give the classification of its representations in terms of the central element. The decomposition of the Hamiltonian realization from Sec. 5 into irreducible q-oscillator representations is developed. Two types of q-oscillator representations appear and, while the Fock representation refers to the bound states, the non-Fock representations cover the continuous spectrum.

In Sec. 7, the consequences of one-dimensional HSUSY are explored for scattering properties [17] of the partner Hamiltonians [18]. The relation between scattering amplitudes of two q-superpartners [10] is presented in Sec. 8. For scattering amplitudes, the “dual” condition of self-similarity (in the momentum space) is defined. An explicit construction satisfying this “dual” self-similarity is presented in terms of a hypergeometric function. This construction corresponds to the reflectionless potential.

In Sec. 9, we analyze other possible realizations of the q-oscillator on the space of wave functions for a one-dimensional Schrödinger Hamiltonian. In our approach, the local Hamiltonian, in general, is not bilinear in creation and annihilation operators but rather belongs to the universal enveloping q-oscillator algebra, i.e., to the algebra of polynomials (analytic functions) of the generators. Thereby the q-oscillator relations are considered as a kind of q-deformed (nonlinear) dynamical algebra. We state a general method of constructing a local Hamiltonian of Schrödinger type with deformed spectrum generating algebras. The related mapping of energy levels is examined. The different forms of the q-oscillator algebra are described, and the constraints on their realization in terms of a Schrödinger Hamiltonian are obtained.

2. SUSY QUANTUM MECHANICS IN ONE DIMENSION

SSQM is generated [5, 6] by the supercharge operators $Q^+$ and $Q^- = (Q^+)^t$, which together with the Hamiltonian $H$ of the system satisfy the relations

\[
(Q^\pm)^2 = 0, \quad [H, Q^\pm] = 0,
\]

\[
\{Q^+, Q^-\} = H = Q^2,
\]

where $Q = Q^+ + Q^-$ is the Hermitian supercharge operator.

The one-dimensional representation is realized by the $2 \times 2$ supercharges

\[
Q^- = \begin{pmatrix} 0 & 0 \\ a^- & 0 \end{pmatrix} \quad \text{and} \quad Q^+ = \begin{pmatrix} 0 & a^+ \\ 0 & 0 \end{pmatrix},
\]

where

\[
a^\pm = \pm \partial + W(x)
\]

and the superHamiltonian is comprised of two ordinary Schrödinger Hamiltonians,

\[
H = \begin{pmatrix} h^{(1)} & 0 \\ 0 & h^{(2)} \end{pmatrix} = \begin{pmatrix} a^+ a^- & 0 \\ 0 & a^- a^+ \end{pmatrix} = -\partial^2 + \begin{pmatrix} W^2 + W' & 0 \\ 0 & W^2 - W' \end{pmatrix} = (-\partial^2 + W^2) 1 + \sigma_3 W'
\]

where $h^{(i)} \equiv -\partial^2 + V^{(i)}(x)$. 

2024