We propose a new method for deriving temporal characteristics of the nonstationary diffusion in a one-dimensional medium without solving the partial differential equation. This method allows the time of relaxation to equilibrium concentration to be determined in a bounded medium with an arbitrary space-varying diffusion coefficient. Some examples are analyzed.

1. INTRODUCTION

Although the mathematical theory of diffusion has been developed in sufficient detail to date (see, e.g., [1-3]), many problems of great theoretical and applied importance remain unsolved. One such problem is that of finding temporal characteristics of the nonstationary diffusion in a medium with a space-varying diffusion coefficient.

The difficulty of solving this problem by the traditional method consists in the necessity of seeking a nonstationary solution of the diffusion equation which is unknown in the general case of a space-varying diffusion coefficient. At the same time, using only numerical methods we cannot find the analytical expression for temporal diffusion characteristics.

Nevertheless, there is a method to avoid this difficulty and to find temporal characteristics of the nonstationary diffusion in an indirect way without solving the partial differential equation. This method is described in the present paper. We will find the time of relaxation to equilibrium concentration of substance in a bounded medium with an arbitrary space-varying diffusion coefficient and consider some examples.

2. FORMULATION OF THE PROBLEM AND METHOD OF SOLUTION

1. Consider a one-dimensional bounded medium \(0 \leq z \leq L\) which has reflecting walls at the ends \((z = 0, z = L)\) and whose diffusion coefficient \(D(z)\) is an arbitrary function of the coordinate which does not become zero nor infinite inside the medium. Let the initial concentration of substance (impurity) of unit mass be localized at one point \(z = z_0\) inside the medium: \(C(z, 0) = \delta(z - z_0)\). Obviously, the impurity will be distributed uniformly in the \([0, L]\) interval with time so that the equilibrium concentration will be equal to \(C(z, \infty) = 1/L\).

We must find the relaxation time — the time of approaching equilibrium concentration at a given point \(z = l\) or in a given layer of the medium \([l_1, l_2]\) (Fig. 1).

2. It is well known that the concentration \(C(z, t)\) satisfies the diffusion equation

\[
\frac{\partial C(z, t)}{\partial t} = \frac{\partial}{\partial z} \left[ D(z) \frac{\partial C(z, t)}{\partial z} \right] = -\frac{\partial G(z, t)}{\partial z},
\]

where

\[
G(z, t) = -D(z) \frac{\partial C(z, t)}{\partial z},
\]
is the flux of substance. The diffusion equation must be supplemented by the following initial and boundary conditions:

\[ C(x,0) = \delta(x - x_0), \quad G(0,t) = G(L,t) = 0. \]

Since the diffusion coefficient \( D(x) \) is assumed to be an arbitrary function of the coordinate, the solution of the diffusion equation (1) is not known, and it does not seem to be possible to determine the relaxation time of the concentration directly through \( C(x,t) \). Nevertheless, it is a problem that can be solved using the new approach based on the Laplace transform for an arbitrary diffusion coefficient.

3. The evolution of \( C(x,t) \) at the observation point \( x = l \) far from the localization point of the initial concentration \( x = x_0 \) takes the form shown in Fig. 2.

We define the relaxation time of substance concentration at \( x = l \) as [4]

\[ \Theta(l) = \int_{0}^{\infty} \frac{[C(l,\infty) - C(l,t)]}{C(l,\infty)} dt, \]  

i.e., through a rectangle of equal area. This definition is valid only if the growing density \( C(l,t) \) does not exceed the equilibrium value \( C(l,\infty) \) and the increase of the density is fast enough to ensure that the integral in (3) converges. If the observation point \( x = l \) is fairly close to \( x_0 \), then the case shown in Fig. 3 can take place in which, according to (3), the definition of the relaxation time is not valid since \( \Theta(l) \) can take not only zero but also negative values.

We now introduce the mass of substance to consider the relaxation time in the layer:

\[ M(t) = \int_{l_1}^{l_2} C(x,t) dx. \]  

(4)

Obviously, for \( x_0 < l_1 < l_2 \leq L \) the initial and final masses are given by \( M(0) = 0 \) and \( M(\infty) = C(x,\infty)(l_2 - l_1) = (l_2 - l_1)/L. \) If the evolution of \( M(t) \) is similar to the function shown in Fig. 2, then we can introduce the relaxation time of substance concentration in the layer as in (3):

\[ \Theta(l_1,l_2) = \int_{0}^{\infty} \frac{[M(\infty) - M(t)]}{M(\infty)} dt/M(\infty). \]  

(5)

It follows from (1) and (4) that

\[ \frac{dM(t)}{dt} = - \int_{l_1}^{l_2} \frac{\partial G(x,t)}{\partial x} dx = -G(l_2,t) + G(l_1,t). \]  

(6)

4. Let us introduce the Laplace transform of the concentration

\[ Y(z,s) = \int_{0}^{\infty} C(z,t)e^{-st} dt. \]  

(7)