Construction of Maximal Sets of Linearly Independent Vector Fields on Spheres

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ABSTRACT. We construct explicitly a maximal set of linearly independent vector fields on any odd-dimensional sphere, i.e., in the general case. The differential geometric properties of the constructed fields are studied. In particular, we find their streamlines, calculate the principal curvatures of the second kind and study holonomic properties of the distributions determined by these fields.

KEY WORDS: orthonormal vector fields on spheres, Adams theorem, distributions, holonomy properties, Frobenius theorem.

We study tangent vector fields over odd-dimensional spheres (it is well known that there are no nonvanishing vector fields on even-dimensional spheres). Below, a vector field on a sphere means a tangent vector field on a sphere. The computer implementation of the earlier approach (see [1, Chap. 11]) to the construction of maximal sets of orthonormal vector fields on spheres meets serious difficulties. This approach associates to a vector field on the sphere $S^{m-1}$ a normalized orthogonal multiplication $\mu : \mathbb{R}^k \times \mathbb{R}^{m-1} \to \mathbb{R}^m$ which is associated, in its own turn, to a representation of the Clifford algebra with $k - 1$ generators in the orthogonal group $O(m)$. This scheme makes it difficult to obtain explicit formulas for orthonormal vector fields on odd-dimensional spheres, and the study of geometric properties of these fields, which have important applications, becomes impossible.

In the present paper we describe a new simpler recurrent way to construct explicitly maximal sets of orthonormal vector fields on spheres of arbitrary odd dimension. This approach allows one to study differential geometric properties of the constructed fields, in particular, their holonomic properties. Until recently, the explicit construction of maximal sets of vector fields on spheres was given only for spheres of dimension $1, 3, 7, 15$ (see [2, p. 187]) (and easily leads to the construction for spheres of dimension $4l - 3, 8l - 5, 16l - 9, 32l - 17$).

The present paper is based on the paper [3], where a class of canonical elliptic systems of the first order is constructed and studied. Using the properties of the characteristic matrices of these systems, we managed to construct explicitly maximal sets of orthonormal vector fields on spheres.

Note that if a maximal set of $M(b)$ orthonormal vector fields on the sphere of dimension $2^b - 1$, where $b$ is a positive integer, is constructed, then it is possible to construct $M(b)$ orthonormal vector fields on any $(l \cdot 2^b - 1)$-dimensional sphere (for $l$ odd); these fields are $l$-fold "copies" of each of the $M(b)$ vector fields on the $(2^b - 1)$-sphere. The Adams Theorem [1] implies that the number $M(b)$ is maximal for the $(l \cdot 2^b - 1)$-sphere as well. It is sufficient, therefore, to construct maximal sets of orthonormal (and, whence, linearly independent) vector fields on $(2^b - 1)$-spheres for all positive integers $b$.

The fields will be constructed recurrently. Let $x = (x_1, \ldots, x_m)$ denote a point in $\mathbb{R}^m$, where $m = 2^b$. For $b = 1, 2, 3$ the required maximal sets were constructed in [3]. The construction can be briefly described as follows. For $b = 1$, we fix a complex structure in $\mathbb{R}^2$ and define the vector field $V_1^1(x)$ by the formula $V_1^1(x) : x \to xi$. For $b = 2$, we fix a quaternion structure in $\mathbb{R}^4$ and define the vector fields $V_1^2(x), V_2^2(x), V_3^2(x)$ by the formulas

$V_1^2(x) : x \to xi, \quad V_2^2(x) : x \to xj, \quad V_3^2(x) : x \to xk.$
For $b = 3$, we fix an octave algebra structure in $\mathbb{R}^8$ and define the vector fields $V_1^3(x), \ldots, V_7^3(x)$ by the formulas

$$
V_1^3(x) : x \rightarrow xi,
V_2^3(x) : x \rightarrow xj,
V_3^3(x) : x \rightarrow xk,
V_4^3(x) : x \rightarrow xE,
V_5^3(x) : x \rightarrow xI,
V_6^3(x) : x \rightarrow xJ,
V_7^3(x) : x \rightarrow xK,
$$

where $i, j, k, E, I, J, K$ are the units of the octave algebra.

Hence the maximal sets of orthonormal vector fields on the $(2b-1)$-spheres for $b = 1, 2, 3$ have been constructed. Now we are going to describe a recurrent procedure for constructing maximal sets of orthonormal vector fields on spheres for the case $b > 3$. Suppose that $8i - 1$ orthonormal vector fields $V_1^{8i-1}(x_1, \ldots, x_m), \ldots, V_{8i-1}^{8i-1}(x_1, \ldots, x_m)$, are constructed, according to the Adams Theorem [1], on $S^{m-1}$, where $m = 2^{4i-1}$ and $b = 4i - 1$, $i$ is a positive integer. Let us construct the maximal set of orthonormal vector fields for $b = 4i, 4i + 1, 4i + 2, 4i + 3 = 4(i + 1) - 1$. Denote by $V_0^b(x_1, \ldots, x_m)$ the set $\{x_1, \ldots, x_m\}$, where $m = 2^b$.

For $b = 4i$, i.e., on $S^{2m-1}$, we construct $8i$ orthonormal vector fields

$$
V_1^{4i}(x_1, \ldots, x_{2m}) = \{V_1^{4i-1}(x_1, \ldots, x_m), -V_1^{4i-1}(x_{m+1}, \ldots, x_{2m})\},
V_2^{4i}(x_1, \ldots, x_{2m}) = \{V_2^{4i-1}(x_1, \ldots, x_m), -V_2^{4i-1}(x_{m+1}, \ldots, x_{2m})\},
V_3^{4i}(x_1, \ldots, x_{2m}) = \{-V_0^{4i-1}(x_{m+1}, \ldots, x_{2m}), V_0^{4i-1}(x_1, \ldots, x_m)\}.
$$

For $b = 4i + 1$, i.e., on $S^{4m-1}$, we construct $8i + 1$ orthonormal vector fields

$$
V_1^{4i+1}(x_1, \ldots, x_{4m}) = \{V_1^{4i}(x_1, \ldots, x_{2m}), -V_1^{4i}(x_{2m+1}, \ldots, x_{4m})\},
V_2^{4i+1}(x_1, \ldots, x_{4m}) = \{V_2^{4i}(x_1, \ldots, x_{2m}), -V_2^{4i}(x_{2m+1}, \ldots, x_{4m})\},
V_3^{4i+1}(x_1, \ldots, x_{4m}) = \{-V_0^{4i}(x_{2m+1}, \ldots, x_{4m}), V_0^{4i}(x_1, \ldots, x_{2m})\}.
$$

For $b = 4i + 2$, i.e., on $S^{8m-1}$, we construct $8i + 3$ orthonormal vector fields

$$
V_1^{4i+2}(x_1, \ldots, x_{8m}) = \{V_1^{4i+1}(x_1, \ldots, x_{4m}), -V_1^{4i+1}(x_{4m+1}, \ldots, x_{8m})\},
V_2^{4i+2}(x_1, \ldots, x_{8m}) = \{V_2^{4i+1}(x_1, \ldots, x_{4m}), -V_2^{4i+1}(x_{4m+1}, \ldots, x_{8m})\},
V_3^{4i+2}(x_1, \ldots, x_{8m}) = \{-V_0^{4i}(x_{4m+1}, \ldots, x_{8m}), V_0^{4i}(x_1, \ldots, x_{4m})\},
V_4^{4i+2}(x_1, \ldots, x_{8m}) = \{-V_0^{4i}(x_{5m+1}, \ldots, x_{8m}), V_0^{4i}(x_1, \ldots, x_{5m})\},
V_5^{4i+2}(x_1, \ldots, x_{8m}) = \{-V_0^{4i}(x_{6m+1}, \ldots, x_{8m}), V_0^{4i}(x_1, \ldots, x_{6m})\},
V_6^{4i+2}(x_1, \ldots, x_{8m}) = \{-V_0^{4i}(x_{7m+1}, \ldots, x_{8m}), V_0^{4i}(x_1, \ldots, x_{7m})\},
V_7^{4i+2}(x_1, \ldots, x_{8m}) = \{-V_0^{4i}(x_{8m+1}, \ldots, x_{8m}), V_0^{4i}(x_1, \ldots, x_{8m})\}.
$$

For $b = 4i + 3$, i.e., on $S^{16m-1}$, we construct $8i + 7$ orthonormal vector fields

$$
V_1^{4i+3}(x_1, \ldots, x_{16m}) = \{V_1^{4i+2}(x_1, \ldots, x_{8m}), -V_1^{4i+2}(x_{8m+1}, \ldots, x_{16m})\},
V_2^{4i+3}(x_1, \ldots, x_{16m}) = \{V_2^{4i+2}(x_1, \ldots, x_{8m}), -V_2^{4i+2}(x_{8m+1}, \ldots, x_{16m})\},
V_3^{4i+3}(x_1, \ldots, x_{16m}) = \{-V_0^{4i+2}(x_{8m+1}, \ldots, x_{16m}), V_0^{4i+2}(x_1, \ldots, x_{8m})\},
V_4^{4i+3}(x_1, \ldots, x_{16m}) = \{-V_0^{4i+2}(x_{9m+1}, \ldots, x_{16m}), V_0^{4i+2}(x_1, \ldots, x_{9m})\},
V_5^{4i+3}(x_1, \ldots, x_{16m}) = \{-V_0^{4i+2}(x_{10m+1}, \ldots, x_{16m}), V_0^{4i+2}(x_1, \ldots, x_{10m})\},
V_6^{4i+3}(x_1, \ldots, x_{16m}) = \{-V_0^{4i+2}(x_{11m+1}, \ldots, x_{16m}), V_0^{4i+2}(x_1, \ldots, x_{11m})\},
V_7^{4i+3}(x_1, \ldots, x_{16m}) = \{-V_0^{4i+2}(x_{12m+1}, \ldots, x_{16m}), V_0^{4i+2}(x_1, \ldots, x_{12m})\},
V_8^{4i+3}(x_1, \ldots, x_{16m}) = \{-V_0^{4i+2}(x_{13m+1}, \ldots, x_{16m}), V_0^{4i+2}(x_1, \ldots, x_{13m})\},
V_9^{4i+3}(x_1, \ldots, x_{16m}) = \{-V_0^{4i+2}(x_{14m+1}, \ldots, x_{16m}), V_0^{4i+2}(x_1, \ldots, x_{14m})\},
V_{10}^{4i+3}(x_1, \ldots, x_{16m}) = \{-V_0^{4i+2}(x_{15m+1}, \ldots, x_{16m}), V_0^{4i+2}(x_1, \ldots, x_{15m})\},
V_{11}^{4i+3}(x_1, \ldots, x_{16m}) = \{-V_0^{4i+2}(x_{16m+1}, \ldots, x_{16m}), V_0^{4i+2}(x_1, \ldots, x_{16m})\}.
$$