Analytic Solutions of the Borel Problem

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ABSTRACT. Let $\mathcal{F} \subset \mathbb{C}$ be a dense-in-itself set that has a nonempty connected interior and contains the origin, and let $C^\infty(\mathcal{F})$ be the space of infinitely differentiable complex-valued functions on $\mathcal{F}$. For some classes of such sets $\mathcal{F}$, we prove that for an arbitrary sequence $\{d_n\}_{n=0}^\infty$ of complex numbers there exists a function $f \in C^\infty(\mathcal{F})$ with $f^{(n)}(0) = d_n$, $n = 0, 1, 2, \ldots$, and study the analyticity properties of $f$. The function $f$ is constructed in the form of various function series, namely, a power series, a series of simple fractions, and an exponential series. Analytic solutions of the multidimensional Borel problem are also considered.

KEY WORDS: Borel problem, analytic solution, $B$-set, domain of existence, singular point, power series, exponential series, simple fraction.

1. Let $\mathcal{F} \subset \mathbb{C}$ be a dense-in-itself set with connected (possibly, empty) interior $\text{int} \mathcal{F}$ such that $0 \in \mathcal{F}$. A complex-valued function $f$ defined on $\mathcal{F}$ is said to be differentiable at some point $z_0 \in \mathcal{F}$ if there exists a finite limit
\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
\]
as $z$ tends to $z_0$ remaining in $\mathcal{F}$. A function $f$ is differentiable on $\mathcal{F}$ if it is differentiable at each point of $\mathcal{F}$. If $f$ is differentiable on $\mathcal{F}$, then $f$ is obviously analytic in the domain $\text{int} \mathcal{F}$. By $C^\infty(\mathcal{F})$ we denote the set of all infinitely differentiable functions on $\mathcal{F}$.

Borel [1] posed the following problem: for a given sequence $\{d_n\}_{n=0}^\infty$ of complex numbers, find a function $y(z) \in C^\infty(\mathcal{F})$ such that
\[
y^{(n)}(0) = d_n, \quad n = 0, 1, 2, \ldots \tag{1}\]
If this problem is solvable in $C^\infty(\mathcal{F})$ for every sequence $\{d_n\}_{n=0}^\infty$, then $\mathcal{F}$ will be called a $B$-set. It is easily seen that if $\mathcal{F}$ is a $B$-set, then $z = 0$ is necessarily a boundary point of $\mathcal{F}$. Indeed, let $0 \in \text{int} \mathcal{F}$.

Consider a sequence $\{d_n\}_{n=0}^\infty$ such that
\[
\lim_{n \to \infty} \left| \frac{d_n}{n!} \right|^{1/n} = \infty. \tag{2}\]
Let $y_0(z)$ be a solution of problem (1) in $C^\infty(\mathcal{F})$. Then $d_n = y_0^{(n)}(0)$ for all $n \geq 0$, and hence
\[
\lim_{n \to \infty} \left| \frac{d_n}{n!} \right|^{1/n} = \lim_{n \to \infty} \left| \frac{y_0^{(n)}(0)}{n!} \right|^{1/n} \leq \frac{1}{\rho(0, \partial \mathcal{F})} < \infty,
\]
where $\rho(0, \partial \mathcal{F})$ is the distance from the point $z = 0$ to the boundary $\partial \mathcal{F}$ of $\mathcal{F}$. This is a contradiction. In the following, we consider dense-in-themselves sets whose interiors are connected and nonempty and whose boundaries contain the point $z = 0$.

It was shown in the original paper [1] that for each sequence $\{d_n\}_{n=0}^\infty$ there exists a function $y(z)$ analytic in the disk $|z - 1| < 1$, infinitely differentiable on the set $|z - 1| \leq 1$, and satisfying relations (1). Since then, the Borel problem has usually been studied in the class $E$, where $E$ is either the space of infinitely differentiable functions on some interval of the real line or on the entire real line, or a subspace of that space (like the Beurling and Roumier spaces; e.g., see [2, 3]). To the space $E$ one naturally assigns some sequence space $H$ such that the sequence $\Pi_f := \{f^{(n)}(0)\}_{n=0}^\infty$ belongs to $H$ for each function $f \in E$;
then the problem is to find conditions under which the operator $\Pi$ is surjective or has a continuous linear right inverse.

To the best of the author’s knowledge, analytic solutions of the Borel problem, explicit expressions for such solutions, and the arrangement of singular points have never been considered.

The same is true of the multidimensional Borel problem, which is also discussed here.

In the present paper, solutions of problem (1) are constructed in the form of various function series, whose structure determines the corresponding $B$-sets. Specifically, we consecutively use power series, series in simple fractions, and exponential series. We study analyticity properties of the solutions (the existence of singular points on the boundary of $F$). The existence of a function representable by the corresponding series and satisfying (1) is established in all cases by the same technique, which is based on the following theorem (e.g., see [4, Chap. 2, Sec. 2.5]).

**Pólya’s Theorem.** Consider the infinite system of linear equations

$$
\sum_{j=1}^{\infty} a_{ij} u_j = b_i, \quad i = 1, 2, \ldots ,
$$

where $\{b_j\}_{j \geq 1}$ is an arbitrary sequence and the coefficients $\{a_{ij}\}$ satisfy the following conditions:

I) the first row $\{a_{1j}\}_{j \geq 1}$ contains infinitely many nonzero entries;

II) $\lim_{j \to \infty} \frac{|a_{ij}| + |a_{ij}| + \cdots + |a_{i-1,j}|}{|a_{ij}|} = 0$, $i = 2, 3, \ldots$.

Then there exists a solution $\{u_j\}_{j = 1}^{\infty}$ of system (3) such that all series on the left-hand side in (3) are absolutely convergent.

As a rule, we seek a solution of problem (1) in the subspace $C^\infty_b(F) \subset C^\infty(F)$ of functions bounded on $F$ together with all of their derivatives. The topology on $C^\infty_b(F)$ is determined by the countable system of norms $\|y\|_n = \sup\{|y^{(k)}(z)| : z \in F, k = 0, 1, \ldots , n\}$, $n = 0, 1, 2, \ldots$, and $C^\infty_b(F)$ equipped with this topology is a Fréchet space.

2. Let $\alpha \in \mathbb{C}$, $\alpha \neq 0$. We set $K_\alpha := \{z : |z - \alpha| < |\alpha|\}$ and $\overline{K_\alpha} := \{z : |z - \alpha| \leq |\alpha|\}$. Next, let $m_n$, $n \geq 1$, be positive integers such that $0 \leq m_1 < m_2 < \cdots$. We seek a solution of problem (1) in $C^\infty(K_\alpha)$ in the form

$$
y(z) = \sum_{n=1}^{\infty} c_n (z - \alpha)^{m_n}.
$$

Consider the auxiliary system

$$
\sum_{n=1}^{\infty} \frac{c_n m_n!}{(m_n - k)!} (-1)^{m_n-k} \alpha^{m_n-k} = d_k, \quad k = 0, 1, 2, \ldots .
$$

We take some $s \geq 2$ and $\varepsilon > 0$. First, suppose that $n$ is so large that $m_n > 2s$. We set

$$
\sigma_{s,n} = \sum_{k=0}^{s-1} \frac{m_n!}{(m_n - k)!} |\alpha|^{m_n-k}, \quad \gamma_{s,n} = \frac{m_n!}{(m_n - s)!} |\alpha|^{m_n-s}, \quad \tau_{s,n} = \frac{\sigma_{s,n}}{\gamma_{s,n}}.
$$

One can readily verify that

$$
\tau_{s,n} < \frac{|\alpha|(|\alpha|^s - 1)}{(m_n - s + 1)(|\alpha| - 1)} \quad \text{for } |\alpha| \neq 1, \quad \tau_{s,n} < \frac{s}{m_n - s + 1} \quad \text{for } |\alpha| = 1.
$$

We choose an $N$ such that $\varepsilon(m_N - s + 1) > \tau_s$, where

$$
\tau_s = \begin{cases} 
\frac{|\alpha|(|\alpha|^s - 1)}{|\alpha| - 1} & \text{for } |\alpha| = 1, \\
\frac{s}{m_n - s + 1} & \text{for } |\alpha| \neq 1.
\end{cases}
$$