Interpolation With Gradient in the Ball and Polydisk

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ABSTRACT. We consider an interpolation problem with gradient for the Banach algebras of bounded holomorphic functions in the ball and polydisk. Sufficient conditions for the solvability of this problem are obtained.

KEY WORDS: Banach algebra, interpolation problem, bounded holomorphic function, interpolation criterion, hyperbolic distance.

§1. Introduction

Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$,
$$\mathbb{B}^n = \{ z = (z^1, \ldots, z^n) \in \mathbb{C}^n : |z|^2 = |z^1|^2 + \cdots + |z^n|^2 < 1 \},$$
and let $\mathbb{D}^n$ be the unit polydisk in $\mathbb{C}^n$,
$$\mathbb{D}^n = \{ z = (z^1, \ldots, z^n) \in \mathbb{C}^n : |z^1| < 1, \ldots, |z^n| < 1 \}.$$
Denote by $\mathcal{D}^n$ any of the domains $\mathbb{B}^n$, $\mathbb{D}^n$ and consider the Banach algebra $H^\infty(\mathcal{D}^n)$ of bounded holomorphic functions in $\mathcal{D}^n$ with the norm
$$\|f\|_{H^\infty} = \sup_{z \in \mathcal{D}^n} |f(z)|, \quad f \in H^\infty(\mathcal{D}^n).$$

Recall that a sequence $\{a_k\}_{k=1}^\infty$ of points in $\mathcal{D}^n$ is interpolating for $H^\infty(\mathcal{D}^n)$ if for any bounded set $\{f_k\}_{k=1}^\infty$ of complex numbers there exists a function $F \in H^\infty(\mathcal{D}^n)$ such that $F(a_k) = f_k$.

Denote by
$$\langle z, \bar{w} \rangle = \sum_{i=1}^n z^i \bar{w}^i$$
the Hermitian inner product in $\mathbb{C}^n$, and by
$$\rho(z, w) = \left(1 - \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, \bar{w} \rangle|^2}\right)^{1/2}$$
the hyperbolic metric in $\mathbb{C}^n$.

In the one-dimensional case Carleson [1] has proved the following interpolation criterion.

**Theorem A.** Let $n = 1$. Then a sequence of points $\{a_k\}_{k=1}^\infty$ in the unit ball $\mathcal{D}^1 = \mathbb{B}^1 = \mathbb{D}^1$ is interpolating for $H^\infty(\mathcal{D}^1)$ if and only if there exist a constant $\delta > 0$ such that
$$\prod_{j : j \neq k} \rho(a_j, a_k) \geq \delta$$
for any $k$ or, equivalently, there exists a constant $C > 0$ such that
$$\sum_{j : j \neq k} (1 - \rho(a_j, a_k)) \leq C$$
for any $k$.

For $n > 1$, the situation is quite different—there is a "gap" between necessary and sufficient interpolation conditions. A sufficient condition for $H^\infty(\mathbb{B}^n)$ proved by Berndtsson [2], is close to the Carleson condition.

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Theorem B. A sequence \( \{a_k \in \mathbb{B}^n\}_{k=1}^{\infty} \) is interpolating for \( H^\infty(\mathbb{B}^n) \) if there exists a constant \( \delta > 0 \) such that
\[
\prod_{j : j \neq k} \rho(a_j, a_k) \geq \delta
\]
for any \( k \).

A sufficient condition for \( H^\infty(\mathbb{D}^n) \), also proved by Berndtsson [3], is similar. Namely, denote by
\[
\rho^s(z, w) = \frac{|z^s - w^s|}{1 - \langle z^s, w^s \rangle}
\]
the hyperbolic distance between the projections of \( z \) and \( w \) onto the \( s \)th coordinate plane in \( \mathbb{C}^n \), and by
\[
\rho^*(z, w) = \max_{1 \leq s \leq n} \rho^s(z, w) = \max_{1 \leq s \leq n} \frac{|z^s - w^s|}{1 - \langle z^s, w^s \rangle}
\]
the polydisk hyperbolic metric in \( \mathbb{C}^n \). Then the Berndtsson's result can be formulated as follows.

Theorem C. A sequence \( \{a_k \in \mathbb{D}^n\}_{k=1}^{\infty} \) is interpolating for \( H^\infty(\mathbb{D}^n) \) if there exists a constant \( \delta > 0 \) such that
\[
\prod_{j : j \neq k} \rho^*(a_j, a_k) \geq \delta
\]
for any \( k \).

However, a necessary condition, due to Varopoulos [4], differs from the sufficient one. Namely, the following theorem is true.

Theorem D. If a sequence \( \{a_k \in \mathbb{B}^n\}_{k=1}^{\infty} \) is interpolating, then there exists a constant \( C > 0 \) such that
\[
\sum_{j : j \neq k} \left( \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - \langle a_k, a_j \rangle|^2} \right)^n \leq C
\]
for any \( k \).

Moreover, Berndtsson has shown in [2] that the exponent \( n \) in the Varopoulos condition is sharp. That is, for any \( \epsilon > 0 \) there exists an interpolating sequence \( \{a_k \in \mathbb{B}^n\}_{k=1}^{\infty} \) such that
\[
\sum_k (1 - |a_k|^2)^{n-\epsilon} = \infty.
\]

One can expect that a "gap" between the necessary and sufficient interpolation conditions may be bridged if one modifies the statement of the interpolation problem. One can try to interpolate, say, not only the values of a function at given points \( \{a_k\} \), but also the values of its gradient at \( \{a_k\} \).

Such an interpolation problem with gradient is considered in this paper. In particular, we prove analogs of Theorems B and C for this problem.

Throughout this paper, \( \partial_s = \partial / \partial z^s \) denotes the derivative with respect to the \( s \)th coordinate and
\[
\partial_m f = \sum_{s=1}^{n} \partial_s f \cdot m^s
\]
denotes the derivative along a vector \( m \).