Arithmetical Properties of the Values of $E$-Functions

A. B. Shidlovskii

ABSTRACT. For a collection of $E$-functions which is algebraically dependent over the field of rational functions, theorems on the algebraic independence of values of subcollections at algebraic points are proved.

Key words: $E$-function, field of rational functions, algebraic dependence, algebraic independence.

For a collection of $E$-functions which is algebraically dependent over $\mathbb{C}(z)$, theorems on the algebraic independence of values of subcollections of the given collection at algebraic points are proved. For the history of the problem, see [1].

Everywhere below we assume that
\begin{equation}
   f_1(z), \ldots, f_m(z)
\end{equation}
is a set of $E$-functions that forms a solution of a system of linear homogeneous differential equations of the form
\begin{equation}
   y_k' = \sum_{i=1}^{m} Q_{k,i} y_i, \quad k = 1, \ldots, m, \quad m \geq 2, \quad Q_{k,i} \in \mathbb{C}(z),
\end{equation}
or of the system
\begin{equation}
   y_k' = Q_{k,0} + \sum_{i=1}^{m} Q_{k,i} y_i, \quad k = 1, \ldots, m, \quad m \geq 2.
\end{equation}

Denote by $T(z) \in \mathbb{C}[z]$ the least common denominator of all functions $Q_{k,i}$ in system (2) or (3), respectively. Furthermore, Let $\xi \in \mathbb{A}$ be such that $\xi T(\xi) \neq 0$, where $\mathbb{A}$ is the field of all algebraic numbers over $\mathbb{Q}$.

As is known, the numbers
\begin{equation}
   f_1(\xi), \ldots, f_m(\xi)
\end{equation}
are homogeneously algebraically independent (algebraically independent) if and only if the functions (1) are homogeneously algebraically independent (algebraically independent) over $\mathbb{C}(z)$, and also that the degrees of homogeneous transcendence (the degrees of transcendence, respectively) of these numbers over $\mathbb{C}(z)$ coincide with those of the functions (1) over $\mathbb{A}$.

We assume now that the degree of homogeneous transcendence (the degree of transcendence) of the functions (1) over $\mathbb{C}(z)$ is equal to $l$, $1 \leq l < m$, and the functions
\begin{equation}
   f_1(z), \ldots, f_l(z)
\end{equation}
are homogeneously algebraically independent (algebraically independent, respectively) over $\mathbb{C}(z)$.

As is known, in this case the numbers
\begin{equation}
   f_1(\xi), \ldots, f_l(\xi)
\end{equation}
are homogeneously algebraically independent (algebraically independent, respectively) for any $\xi \in \mathbb{A}$ except for finitely many numbers of this form. Certain theorems on the algebraic independence of the numbers (6) are established for some subclasses of $E$-functions and for exactly specified points $\xi$ in [1].

Original article submitted January 12, 1999.
In what follows, we assume that, in the polynomials belonging to \( \mathbb{C}[z, z_1, \ldots, z_m] \) and \( \mathbb{C}[z_1, \ldots, z_m] \), the terms are ordered lexicographically with respect to the degrees of \( z_m, \ldots, z_1 \) and their coefficients belong to \( \mathbb{C}[z] \) and \( \mathbb{C} \), respectively.

Consider a collection of homogeneous (arbitrary, respectively) minimal equations of the functions (1) over \( \mathbb{C}[z] \) (see [2] or [1, Chap. 4, §10]). Let \( \Lambda \) be the set of zeros of all their leading coefficients.

Let \( K \) be an arbitrary chosen algebraic field over \( \mathbb{Q} \) that contains the coefficients of the series in powers of \( z \) of all functions (1), let \( \xi \in \Lambda \) be a number such that \( \xi T(\xi) \neq 0 \), and let \( h = [K : \mathbb{Q}] \). Further, let \( K_1, \ldots, K_h \) be the algebraic fields conjugate to \( K \), let \( \xi_i \in K_i \), \( i = 1, \ldots, h \), be the numbers conjugate to \( \xi \), let \( f_{1,i}(z), \ldots, f_{m,i}(z) \) be the functions obtained from those in (1) by replacing all coefficients of their power series by the corresponding conjugate numbers of the field \( K_i \), and let the polynomials \( P_i, Q_i, R_i \), \( i = 1, \ldots, h \), be similarly obtained from the polynomials \( P, Q, R \in \mathbb{K}[z, z_1, \ldots, z_m] \) introduced below.

In 1973, V. G. Chirskii proved the following theorem in the inhomogeneous case [3].

**Theorem 1.** If the coefficients of the power series of the functions (1) and the number \( \xi \), \( \xi T(\xi) \neq 0 \), \( \xi \not\in \Lambda \), belong to an imaginary quadratic algebraic field over \( \mathbb{Q} \), then the numbers (6) are homogeneously algebraically independent (algebraically independent, respectively).

Under similar assumptions, but for an arbitrary field \( K \), the following assertion was established in 1989 (see [2]).

**Theorem 2.** There is an index \( i \), \( 1 \leq i \leq h \), such that the numbers
\[
(7)
\]
are homogeneously algebraically independent (algebraically independent, respectively).

Below we prove an assertion that generalizes Theorems 1 and 2.

Let \( \xi \in \mathbb{C} \) and let \( \varphi : \mathbb{C}[z, z_1, \ldots, z_m] \to \mathbb{C}[z_1, \ldots, z_m] \) be the homomorphism obtained by replacing the variable \( z \) by \( \xi \) in all polynomials of the ring \( \mathbb{C}[z, z_1, \ldots, z_m] \). For an arbitrary ideal \( \mathfrak{U} \) of this ring, we denote by \( \mathfrak{U}_\xi = \varphi \mathfrak{U} \mathfrak{U} \) the ideal corresponding to \( \mathfrak{U} \) in the ring \( \mathbb{C}[z_1, \ldots, z_m] \) under the homomorphism \( \varphi \).

Let \( \mathfrak{P}_\xi^0 \) (\( \mathfrak{P} \), respectively) be the prime ideal of the ring \( \mathbb{C}[z, z_1, \ldots, z_m] \) generated by all polynomials homogeneous with respect to \( z_1, \ldots, z_m \) (by all polynomials, respectively) that vanish when the functions (1) are substituted for \( z_1, \ldots, z_m \). If the functions (1) are homogeneously algebraically independent, then we set \( \mathfrak{P}_\xi^0 = (0) \) (\( \mathfrak{P} = (0) \)).

Note that the ideal \( \mathfrak{P}_\xi^0 \) is prime for any \( \xi \in \Lambda \) except for finitely many such numbers [4].

If \( \xi \in \Lambda \) satisfies \( \xi T(\xi) \neq 0 \) and the homogeneous ideal \( \mathfrak{P}_\xi^0 \) (the ideal \( \mathfrak{P}_\xi \)) corresponding to the functions (1) is prime, then the collection of the functions (1) is said to be homogeneously prime (prime, respectively) at the point \( \xi \).

**Theorem 3.** Assume that a collection of \( E \)-functions (1) forms a solution of system (2) (of system (3), respectively), has the degree of homogeneous transcendence (the degree of transcendence, respectively) over \( \mathbb{C}(z) \) equal to \( l \), \( 1 \leq l \leq m \), and is homogeneously prime (prime, respectively) at a point \( \xi \in \Lambda \), such that \( \xi T(\xi) \neq 0 \) and \( \xi \not\in \Lambda \). We also assume that the functions (5) are homogeneously algebraically independent (algebraically independent, respectively) over \( \mathbb{C}(z) \).

Then the numbers (7) are homogeneously algebraically independent (algebraically independent, respectively) for any \( i \), \( 1 \leq i \leq h \).

We note that the inhomogeneous versions of Theorems 1, 2, and 3 are consequences of their homogeneous versions. We must only replace the number \( m \) by \( m+1 \) and set \( f_{m+1}(z) \equiv 1 \). Therefore, in the subsequent discussion, we consider the homogeneous cases only.

To prove Theorem 3, we shall need two theorems that are of independent interest as well.

**Theorem 4** [4]. If a collection (1) of \( E \)-functions forms a solution of system (3) and is homogeneously prime at a point \( \xi \in \Lambda \), \( \xi T(\xi) \neq 0 \), then, for any homogeneous polynomial
\[
P = P(z_1, \ldots, z_m) \in \mathbb{A}[z_1, \ldots, z_m], \quad P \not\in \mathfrak{P}_\xi^0,
\]
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