RESEARCH ARTICLE

The Pseudovariety Generated by Completely 0-Simple Semigroups

G. Mashevitzky*

Communicated by Boris M. Schein

Abstract

A finite basis of pseudoidentities of the pseudovariety generated by all finite completely 0-simple semigroups is constructed. Thus this pseudovariety is decidable.

Introduction

A pseudovariety of semigroups is a class of finite semigroups that is closed with respect to subalgebras, homomorphic images and finite direct products. It is obvious that the trace $\mathcal{V} \cap \mathcal{F}$ of any variety $\mathcal{V}$ of semigroups in the class $\mathcal{F}$ of all finite semigroups is a pseudovariety. Any pseudovariety is such a trace or a direct limit of a direct family of such traces [2].

An $n$-ary implicit operation on a pseudovariety $\mathcal{P}$ is a mapping $\pi$ associating with each finite semigroup $S \in \mathcal{P}$ an $n$-ary function $\pi_S : S^n \to S$ on $S$ such that, for every homomorphism $f : S \to T$ between finite semigroups from $\mathcal{P}$, $\pi$ commutes with $f$, that is

$$(\forall s_1, \ldots, s_n \in S)(f(\pi_S(s_1, \ldots, s_n)) = \pi_T(f(s_1), \ldots, f(s_n))$$

A pseudoidentity is a formal identity of implicit operations. A pseudoidentity $\pi = \rho$ is valid in a semigroup $S$ if $\pi_S = \rho_S$. Every pseudovariety can be defined by a system of pseudoidentities, that is, it consists of all finite semigroups satisfying the given set of pseudoidentities [8].

Let $S$ be a finite semigroup. For every element $a \in S$, in the monogenic subsemigroup $(a)$ there exists a unique idempotent $a^w$. The function $a \to a^w$ is an example of an implicit operation.

One of the central problems in the theory of pseudovarieties is the membership problem: does a given finite semigroup belong to a given pseudovariety? The membership problem is decidable if there exists an algorithm that solves this problem. A pseudovariety with decidable membership problem is called decidable.

Let $\Xi_0$ denote the system of the following four pseudoidentities:

(1) $$(xy)^{w+1}x = xyx$$

(2) $$x^{w+2} = x^2$$

* Partially supported by Israel Ministry of Absorption
(3) \[(xyz)^w xhz = xhz(xyz)^w\]

(4) \[(xy)^w xzx = xz(xy)^w x.\]

Let \( PG \) be a pseudovariety of groups and \( P\text{Var}(CS^0(PG)) \) the pseudovariety generated by all finite completely 0-simple semigroups over the groups from \( PG \).

We prove the following theorem and its corollaries.

**Theorem.** Let \( \Xi_1 \) be a system of pseudoidentities holding in the pseudovariety \( P\text{Var}(CS^0(PG)) \) and suppose that it forms a basis of pseudoidentities of the pseudovariety \( PG \) in the class of groups (that is, it distinguishes \( PG \) among pseudovarieties of groups). Then the system of pseudoidentities \( \Xi' = \Xi_1 \cup \Xi_0 \) forms a basis of pseudoidentities of the pseudovariety \( P\text{Var}(CS^0(PG)) \).

**Corollary 1.** Let \( PG \) be a decidable pseudovariety of groups. Then the pseudovariety \( P\text{Var}(CS^0(PG)) \) is decidable.

**Corollary 2.** \( \Xi_0 \) is a basis of the pseudovariety generated by all finite completely 0-simple semigroups.

**Corollary 3.** The pseudovariety generated by all finite completely 0-simple semigroups is decidable.

Our proof uses an improved and developed version of a method that reduces the finite basis problem in the class of all semigroups to the same problem in the class of regular semigroups (see [6] and [7]).

An independent and alternative proof of these Corollaries has been announced in [3].

1. **Notation and Preliminaries**

The facts of the semigroup theory that we use can be found in [1] and [4].

Let \( G \) be a group, \( G^0 \) the semigroup \( G \cup \{0\} \), \( I \) and \( \Lambda \) sets and \( P \) an \( I \times \Lambda \) matrix over \( G^0 \). Let \( M^0(G,I,P,\Lambda) \) be the set \( \{(a,\lambda,i)|a \in G, \lambda \in \Lambda, i \in I\} \cup \{0\} \) with the multiplication

\[ (a,\lambda,i)(b,\mu,j) = \begin{cases} (ap_{iu}b,\lambda,j), & \text{if } p_{iu} \neq 0, \\
0, & \text{if } p_{iu} = 0. \end{cases} \]

0 is a zero of this semigroup.

This defines a semigroup called a **Rees semigroup of matrix type** over \( G \) presented with the help of the sandwich matrix \( P \). The matrix \( P \) is called **regular** if there are nonzero elements in every row and in every column of \( P \). A semigroup is completely 0-simple if and only if it is isomorphic to a Rees semigroup of matrix type over a group with 0 adjoined presented with the help of a regular sandwich matrix ([1] and [4]).

Let \( G \) be a variety of groups. Denote by \( CS^0(G) \) the class of all completely 0-simple semigroups over groups from \( G \). Let \( CS^0_k \) denote the class of all completely 0-simple semigroups over groups of exponent \( k \).