On Perturbations of Differentiable Semigroups

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Abstract

Let $X$ be a Banach space and let $A$ be the infinitesimal generator of a differentiable semigroup $\{T(t) \mid t \geq 0\}$, i.e. a $C_0$-semigroup such that $t \mapsto T(t)x$ is differentiable on $(0, \infty)$ for every $x \in X$. Let $B$ be a bounded linear operator on $X$ and let $\{S(t) \mid t \geq 0\}$ be the semigroup generated by $A + B$. Renardy recently gave an example which shows that $\{S(t) \mid t \geq 0\}$ need not be differentiable. In this paper we give a condition on the growth of $\|T'(t)\|$ as $t \downarrow 0$ which is sufficient to ensure that $\{S(t) \mid t \geq 0\}$ is differentiable. Moreover, we use Renardy's example to study the optimality of our growth condition. Our results can be summarized roughly as follows:

(i) If $\limsup_{t \to 0^+} \frac{t \log \|T'(t)\|}{\log(t)} = 0$ then $\{S(t) \mid t \geq 0\}$ is differentiable.

(ii) If $0 < L = \limsup_{t \to 0^+} \frac{t \log \|T'(t)\|}{\log(t)} < \infty$ then $t \mapsto S(t)$ is differentiable on $(L, \infty)$ in the uniform operator topology, but need not be differentiable near zero.

(iii) For each function $\alpha : (0, 1) \to (0, \infty)$ with $\alpha(t)/\log(1/t) \to \infty$ as $t \downarrow 0$, Renardy's example can be adjusted so that $\limsup_{t \to 0^+} \frac{t \log \|T'(t)\|}{\alpha(t)} = 0$ and $t \mapsto S(t)$ is nowhere differentiable on $(0, \infty)$.

We also show that if $\limsup_{t \to 0^+} t^p \|T'(t)\| < \infty$ for a given $p \in [1, \infty)$, then $\limsup_{t \to 0^+} t^p \|S'(t)\| < \infty$; it was known previously that if $\limsup_{t \to 0^+} t^p \|T'(t)\| < \infty$, then $\{S(t) \mid t \geq 0\}$ is differentiable and $\limsup_{t \to 0^+} t^{2p-1} \|S'(t)\| < \infty$.

1. Introduction and Statement of Results

Let $(X, \|\cdot\|)$ be a complex Banach space and let $\{T(t) \mid t \geq 0\}$ be a linear $C_0$-semigroup on $X$ with infinitesimal generator $A$. We say that $\{T(t) \mid t \geq 0\}$ is differentiable if the mapping $t \mapsto T(t)x$ is differentiable on $(0, \infty)$ for every $x \in X$; we say that $\{T(t) \mid t \geq 0\}$ is eventually differentiable if there exists $t_0 \in (0, \infty)$ such that $t \mapsto T(t)x$ is differentiable on $(t_0, \infty)$ for every $x \in X$. It is well known that $\{T(t) \mid t \geq 0\}$ is differentiable if and only if $T(t)$ maps $X$ into $D(A)$, the domain of $A$, for every $t > 0$. Moreover, if $\{T(t) \mid t \geq 0\}$ is differentiable then $t \mapsto T(t)$ is of class $C^\infty$ on $(0, \infty)$ in the uniform operator topology, $T(t)$ maps $X$ into $\cap_{n=1}^\infty D(A^n)$ for every $t > 0$, and

$$T^{(n)}(t) = A^n T(t) = (AT(\frac{t}{n}))^n = (T(\frac{t}{n}))^n$$

for every $n = 1, 2, 3, \ldots$ and every $t > 0$. We note also that if $\{T(t) \mid t \geq 0\}$ is differentiable and $A$ is unbounded then $\|T'(t)\| = \|AT(t)\| \to \infty$ as $t \downarrow 0$; there is no upper limit to the rate of growth of $\|AT(t)\|$ as $t \downarrow 0$. We refer to [4] and Section
2.4 of [6] for more information on the basic properties of differentiable and eventually differentiable semigroups.

Let $B$ be a bounded linear operator on $X$. Then $A + B$ generates a $C_0$-semigroup $\{S(t) \mid t \geq 0 \}$. It is of interest to know which properties of $\{T(t) \mid t \geq 0 \}$ are inherited by $\{S(t) \mid t \geq 0 \}$. In 1953, Phillips [7] gave an example of a $C_0$-semigroup $\{T(t) \mid t \geq 0 \}$ and a special choice of $B$ such that $T(t) = 0$ for all $t \geq 1$ and yet there is no $t_0 \in [0, \infty)$ at which the mapping $t \mapsto S(t)$ is continuous in the uniform operator topology. Such a semigroup $\{S(t) \mid t \geq 0 \}$ cannot be eventually differentiable; moreover, there can be no $t_0 \in [0, \infty)$ such that $S(t_0)$ is compact.

Roughly speaking, Phillips' example shows that one should not expect eventual regularity of $\{T(t) \mid t \geq 0 \}$ to be inherited by $\{S(t) \mid t \geq 0 \}$. On the other hand it is known that if $\{T(t) \mid t \geq 0 \}$ is an analytic semigroup, or if $T(t)$ is compact for every $t > 0$, or if $t \mapsto T(t)$ is continuous on $(0, \infty)$ in the uniform topology, then $\{S(t) \mid t \geq 0 \}$ must also have the corresponding property ([2], [5], [7]). In 1968, Pazy [5] showed that a certain decay condition on the resolvent of $A$ (that was already known to imply differentiability of $\{T(t) \mid t \geq 0 \}$) is sufficient to ensure differentiability of $\{S(t) \mid t \geq 0 \}$ for every bounded linear $B$. Pazy also raised the question of whether or not differentiability of $\{T(t) \mid t \geq 0 \}$ alone is enough to imply differentiability of $\{S(t) \mid t \geq 0 \}$. Renardy [8] recently gave a negative answer to this question by constructing a counterexample on $l^2$.

In this paper we show that if $\{T(t) \mid t \geq 0 \}$ is differentiable and $\|AT(t)\|$ obeys a certain growth condition as $t \downarrow 0$ then $\{S(t) \mid t \geq 0 \}$ is differentiable for every bounded linear $B$. We show further that no weaker growth condition is sufficient to ensure the same conclusion, and we discuss in detail the special case when $\|AT(t)\|$ grows exactly at a certain critical level. In addition, we make an observation concerning a special class of differentiable semigroups discussed by Crandall & Pazy [1]. This class is described in the following definition.

**Definition 1.** Let $\{T(t) \mid t \geq 0 \}$ be a $C_0$-semigroup with infinitesimal generator $A$ and let $p \in [1, \infty)$ be given. We say that $\{T(t) \mid t \geq 0 \}$ is of class $D^p$ if it is differentiable and

$$\limsup_{t \to 0^+} t^p \|AT(t)\| < \infty.$$

We note that $D^1$ is precisely the class of analytic semigroups.

The remark below summarizes several earlier results that are relevant to the present discussion. We write $R(\lambda; A) = (\lambda I - A)^{-1}$ for all $\lambda \in \rho(A)$, the resolvent set of $A$.

**Remark 1.** Let $\{T(t) \mid t \geq 0 \}$ be a $C_0$-semigroup with infinitesimal generator $A$ and choose $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Let $B$ be a bounded linear operator on $X$ and let $\{S(t) \mid t \geq 0 \}$ be the $C_0$-semigroup generated by $A + B$.

a) Pazy [5] showed that if

$$\lim_{|\tau| \to \infty} \log |\tau| \cdot \|R(\mu + i\tau; A)\| = 0$$

for some real $\mu \geq \omega$