ARBORICITY AND COMPLEMENT OF A GRAPH*

WANG JIANFANG (王建方)
(Institute of Applied Mathematics, the Chinese Academy of Sciences
and
Asia-Pacific Operational Research Center, Beijing 100080, China)

CHEN CHUANPING (陈传平)
(Institute of System Science, the Chinese Academy of Sciences, Beijing 100080, China)

ZHANG ZHONGFU (张忠辅)
(Department of Basic Courses, Lanzhou Railway Institute, Lanzhou 730070, China)

Abstract

The arboricity of graph $G=(V,E)$, denoted by $a(G)$, is defined as

$$a(G)=\min\{n\mid E \text{ can be partitioned into } n \text{ subsets } E_1, E_2, \ldots, E_n, \text{ such that each subset spans a subgraph of } G \text{ so as to be a forest}\}.$$ 

In this paper the following results have been obtained. For any graph $G$ of order $p$,

$$a(G)+a(\overline{G}) \leq \begin{cases} \left\lfloor \frac{5p+4}{8} \right\rfloor & \text{if } p \leq 4, \\ \left\lfloor \frac{5p+5}{8} \right\rfloor & \text{if } p = 7, \\ \left\lfloor \frac{5p+6}{8} \right\rfloor & \text{if } p = 10, \\ \left\lfloor \frac{5p+7}{8} \right\rfloor & \text{for other cases}, \end{cases}$$

and the bounds are sharp; especially as an integer function, $5p+7$ could not be decreased. Furthermore, Nordhaus-Gaddum Theorem for arboricity has also been got.

Key words. Arboricity, complement, vertex-arboricity

1. Introduction

All the graphs we consider are undirected and simple. The complete graph of order $n$ is denoted by $K_n$, an independent set of $n$ vertices by $\overline{K}_n$. The notation $G \cup H$ means

Received May 24, 1994.
* This research is supported by the National Natural Science Foundation of China.
the disjoint union of two graphs G and H, and G + H the disjoint union of G and H plus all the edges between G and H. H ≤ G indicates that H is a subgraph of G. \( \alpha(G) \) is the independent number of G, \( \theta(G) \) the cardinal of the maximum clique of G.

The arboricity of a graph G is the minimum number of subsets into which the edge set of G can be partitioned such that each subsets spans a subgraph of G so as to be forest. The vertex-arboricity of G is the minimum number of subsets into which the vertex set of G can be partitioned such that each induced subgraph by each subset is a forest. The arboricity and vertex-arboricity of G are denoted by \( a(G) \) and \( \rho(G) \) respectively.

Arboricity is an important invariant in graph theory. More than 30 years ago, Nash-Williams got a well-known theorem for arboricity.

**Theorem A** \([1]\). For any graph G of order \( p \geq 2 \),

\[
a(G) = \max \left\lfloor \frac{q_n}{n-1} \right\rfloor,
\]

where \( q_n = \max \{|E(H)| \mid H \text{ is a subgraph of } G \text{ with order } n \} \).

\( \left\lfloor X \right\rfloor \) and \( \left\lceil X \right\rceil \) denote the maximum integer which is not greater than X and the minimum integer which is not less than X, respectively.

Obviously, in general, complexity for \( \max \frac{q_n}{n-1} \) is not polynomial.

In 1970, Mitchem obtained Nordhaus-Gaddum type theorem for vertex-arboricity.

**Theorem B** \([3]\). For any graph G of order \( p \geq 2 \),

\[
\sqrt{p} \leq \rho(G) + \rho(\overline{G}) \leq 1 + \frac{p+1}{2}, \quad \left\lceil \frac{p}{4} \right\rceil \rho(G) \cdot \rho(\overline{G}) \leq \left( \frac{p+3}{4} \right)^2.
\]

In \([4]\) there is a conjecture which is the same as the first inequality in the theorem. Here we would like to tell it to our friends of the same occupation.

Zhang et al\([2]\) studied the relation between the arboricities of a graph and its complement graph and got that for a graph of order \( p \geq 2 \),

\[
\left\lceil \frac{p}{2} \right\rceil \leq a(G) + a(\overline{G}) \leq p.
\] (1)

Gao and Xu\([5]\) obtained an improved upper bound for \( a(G) + a(\overline{G}) \): For any graph G of order \( p \geq 4 \),

\[
a(G) + a(\overline{G}) \leq p - 1.
\] (2)

It is easily seen that (1) does not hold for \( p = 1 \), the upper bound in (1) can be reached when \( p = 2, 3 \). Similarly, when \( p \leq 3 \), (2) does not hold, can be reached for \( p = 4, 5, 6 \). But for \( p \geq 7 \), the bound in (2) could not be attained and the difference between the bound and \( a(G) + a(\overline{G}) \to \infty \) when \( p \to \infty \).

In this paper, we will obtain a new upper bound for \( a(G) + a(\overline{G}) \) which can be reached for any positive integer \( p \), and get the Nordhaus-Gaddum theorem for arboricity.

2. Results

**Lemma 1** \([6]\). For any graph \( \rho(G) \leq a(G) \).

Let \( G_{p,t} = K_t \cup \overline{K}_{p-t} \), where \( t < p \).

**Lemma 2.** If \( 2n - 1 < p \), then

\[
a(\overline{G}_{p,2n-1}) = \left\lceil \frac{p}{2} - \frac{(2n-1)(n-1)}{p-1} \right\rceil, \quad a(\overline{G}_{p,2n}) = \left\lceil \frac{p}{2} - \frac{n(2n-1)}{p-1} \right\rceil.
\]