SIMPLE PROOF OF A LARGE DEVIATION RESULT*

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Abstract

In this note we prove a large deviation result for a perturbed discontinuous inhomogeneous system which complements [1].

Key words. Large deviation principle, stochastic differential equation, contraction principle, Donsker's invariance principle

In this note we prove the large deviation principle for the following process in $\mathbb{R}$:

$$dx_t = b(x_t, t)dt + \varepsilon dw_t, \quad t \in [0, 1],$$

where the initial point $x_0$ is fixed, $w_t$ is a Wiener process, $\varepsilon > 0$ is a small parameter which tends to 0, $b(x, t)$ is a bounded piecewise Lipschitz function of the form

$$b(x, t) = \begin{cases} 
    b_+(x, t), & x > \theta_t, \\
    b_-(x, t), & x < \theta_t,
\end{cases}$$

where $\theta$ is a smooth curve, $b_\pm$ are two bounded Lipschitz functions on $(-\infty, \infty) \times [0, \infty)$ satisfying

$$b_-(\theta_t, t) \geq b_+(\theta_t, t).$$

If (3) is replaced with an opposite inequality, the large deviation results are proved in [2] and [1], for homogeneous and inhomogeneous cases respectively. In case $\theta_t \equiv 0$ and $b_-(0, t) - b_+(0, t) \geq c$ for some constant $c > 0$, Korostelev and Leonov proved in [3,4] the large deviation result (even with the state space $\mathbb{R}^2$). Here we relieve their assumption of uniform positivity of the jump of $b$ on the discontinuity line $\theta$.

Recently Chiang and Sheu [5] obtained more general large deviation results. Their proof used more complicated tools from stochastic analysis, including local time. The proof in this note is somewhat elementary, and it demonstrates an application of Donsker's celebrated invariance principle to large deviation theory.

First we recall the notion of large deviation principle.

Let $\{Q^\varepsilon, \varepsilon > 0\}$ be a family of probability measures on a metric space $E$. We say $Q^\varepsilon$ satisfies the large deviation principle as $\varepsilon$ tends to 0, if there exists a $[0, \infty]$-valued functional $I$ on $E$ satisfying the following conditions:

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The functional $I$ is called the rate functional.

In this paper $E = C[0, 1]$, the space of continuous functions on $[0, 1]$, with the topology generated by the sup norm.

The main result is the following theorem:

**Theorem 1.** Suppose that $(x_t)$ is determined by (1) with $b(x, t)$ as in (2) and (3). Then $(x_t)$ satisfies the large deviation principle as $\varepsilon$ tends to 0, with rate functional

$$J(\psi) = \begin{cases} 
\inf \left\{ \frac{1}{2} \int_0^1 |\psi_t - [\alpha_t b_+(\psi_t, t) + (1 - \alpha_t) b_-(\psi_t, t)]|^2 dt, \right. \\
\left. \quad \alpha : [0, 1] \to [0, 1] \text{ is Borel measurable}, \\
\quad \alpha_t = 1 \text{ if } \psi_t > \theta_t, \quad \alpha_t = 0 \text{ if } \psi_t < \theta_t \}, \right. \\
\quad \text{if } \psi \text{ is absolutely continuous and } \psi_0 = x_0; \right. \\
\quad +\infty, \quad \text{otherwise}. \right. \tag{4}
$$

The main tool we use is the following contraction principle.

**Lemma 0 (Contraction Principle. See [6, Theorem 3.3.1]).** Suppose $f$ is a continuous mapping from one metric space $S_1$ to $S_2$, another metric space. Let $\mu_\varepsilon$, $\varepsilon > 0$ be a family of probability measures on $S_1$, satisfying the large deviation principle with rate functional $I_1$. Then $\nu_\varepsilon = \mu_\varepsilon \cdot f^{-1}$, the measure on $S_2$ induced by $f$, satisfies the large deviation principle with rate functional $I_2(\psi) = \inf \{ I_1(\phi) : f(\phi) = \psi \}$.

For the time being we suppose that $\theta_t \equiv 0$.

Following [4], denote

$$L = \{ \gamma \in C[0, 1] : \gamma \text{ is absolutely continuous with initial value} \}
$$

$$\gamma_0 = 0, \ |\dot{x}_t| > 2\|b\| \text{ for almost all } t),$$

where $\|b\|$ is the sup norm of $b$. It is well known that $L$ is dense in $C_0[0, 1]$, the class of continuous functions with initial value 0.

For $\phi \in L$, let $G(\phi)$ be a solution to the following equation

$$\dot{\psi}_t = x_0 + \int_0^t b(\psi_s, s) \, ds + \phi_t, \tag{5}$$

where $x_0$ is the same initial position as in (1).

Using Tonelli’s approximation one easily see the existence. In fact, for $n \in N$, define $\psi^n_t = x_0, \text{ on } [0, 1/n]; \psi^n_t = x_0 + \int_0^{t-1/n} b(\psi^n_s, s) \, ds + \phi_{t-1/n}$ for $t > 1/n$. Since $b$ is bounded and $\phi$ is absolutely continuous, $\psi^n, \ n = 1, 2, \ldots$ is uniformly bounded and equicontinuous. According to Ascoli-Arzela Lemma, there is a subsequence of $\{\psi^n\}$ converging to some $\psi \in C[0, 1]$. It is not hard to see that $\psi$ satisfies (5) (see also the proof of Lemma 3 below). On the other hand the uniqueness is guaranteed by the following lemma. Hence $G$ is a mapping.