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A Based Federer Spectral Sequence and the Rational Homotopy of Function Spaces

Samuel B. Smith
Department of Mathematics
Saint Joseph's University
Philadelphia, PA 19131

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We study the rational homotopy of function spaces within the context of Quillen's minimal models. Our method is to consider a spectral sequence with $E_2^{p,q} = \tilde{H}^q(X, \pi_{p+q}(Y) \otimes \mathbb{Q})$ converging to the rational homotopy groups of components of the based function space $M(X,Y)_*$. Our results include calculations of rational homotopy groups as well as general contributions to the rational classification problem for components of function spaces.

1. An Exact Couple. Let $M(X,Y)$ and $M(X,Y)_*$ denote, respectively, the spaces of free and based continuous functions between two spaces X and Y. As an overriding assumption, all spaces are taken to be simply connected CW complexes. We denote the path components corresponding to a map $f : X \to Y$ by $M_f(X,Y)$ and $M_f(X,Y)_*$.

The based Federer spectral sequence for a map $f : X \to Y$ arises, like the original [1], from an exact couple of the form

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & C \arrow[ul, "k"] & \end{tikzcd}$$

Suppose X comes equipped with a fixed CW decomposition so

that for each $q > 1$ there is a cofibration sequence

$$\bigvee_{\alpha} S_{\alpha}^{q-1} \xrightarrow{h_q} X^{q-1} \to X^q,$$

where X^q is the q-skeleton of X and h_q is the wedge of the attaching maps for the q-cells of X. Let $f_q : X^q \to Y$ denote the restriction of f and let $W^{q-1} = \bigvee_\alpha S_\alpha^{q-1}$. We obtain a long exact sequence on homotopy of which a portion is

$$\pi_{p+1}(M_0(W^{q-1}, Y)_*) \xrightarrow{\partial_q} \pi_p(M_{f_q}(X^q, Y)_*) \xrightarrow{(\rho_q)_*}$$

$$\pi_p(M_{f_{q-1}}(X^{q-1}, Y)_*) \xrightarrow{(\overline{h_q})_*} \pi_p(M_0(W^{q-1}, Y)_*).$$

Set $A_{p,q} = \pi_p(M_{f_q}(X^q, Y)_*)$, $C_{p,q} = \pi_{p+1}(M_0(W^{q-1}, Y)_*)$, and let $i = (\rho_q)_*$, $j = (\overline{h_q})_*$ and $k = \partial_q$. By adjointness, $C_{p,q} \cong \bigoplus_\alpha \pi_{p+1}(\Omega^{q-1}(Y)) \cong \widetilde{C}^q(X, \pi_{p+q}(Y))$, the reduced cellular q-cochains of X. When X is a finite complex, by [6, Theorem 2.3] we may replace X by any rationally equivalent space without affecting the rational homotopy of $M_f(X, Y)_*$. In particular, we may assume X comes equipped with a minimal CW decomposition with respect to its rational homotopy type. In this case, for degree reasons there are no nontrivial coboundary relations in the cellular cochain complex for X and so $C_{p,q} = \widetilde{H}^q(X, \pi_{p+q}(Y))$. Thus we have

Theorem 1.1 *Let X be finite and $f : X \to Y$ a based map. Then there is a spectral sequence with $E_{p,q}^2 = \widetilde{H}^q(X, \pi_{p+q}(Y) \otimes \mathbb{Q})$ converging to $\pi_p(M_f(X, Y)_*) \otimes \mathbb{Q}$.* □

2. Null Components. We prove that the spectral sequence collapses on null components. The key lemma here is

Lemma 2.1 *Let X have dimension n and suppose $\alpha \in \pi_m(X)$ for $m > n$. Then $\Sigma(\alpha)$ is of finite order in $\pi_{m+1}(\Sigma(X))$.*

Proof. Let $f : S^m \to X$ represent α and let T_f denote the mapping cone of f. Since $\Sigma(T_f) \simeq T_{\Sigma f}$ it follows that $\Sigma(T_f)$ admits a minimal CW decomposition for which Σf is the attaching map for the top cell. But $\Sigma(T_f)$ is rationally equivalent to a wedge