NONLOCAL NONLINEAR INTERACTION OF HF WAVES WITH THE IONOSPHERE

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We analyze the behavior of an HF wave beam near the peak of the F2 layer of the ionosphere taking into account the electron diffusion. The formulation of the problem is presented and its numerical solution is obtained in a physically reasonable approximation.

In this paper, we analyze the solution of the nonlinear Helmholtz equation. The considered solution is localized near a curve that has the meaning of a ray path. The present paper continues the studies that were reported in [1].

In contrast to the linear problem on diffraction, the ray path in the nonlinear problem depends on the solution [2], and deriving the ray path is a separate problem. The so-called quasi-ray coordinates \( \xi \) and \( \eta \) are introduced in the vicinity of the selected ray path. The coordinate \( \xi \) is the length of the path, and \( \eta \) is the distance along the normal to the curve. The Helmholtz equation in terms of these coordinates has the following form:

\[
\frac{\partial}{\partial \xi} \left( \frac{1}{\beta} \frac{\partial u}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \beta \frac{\partial u}{\partial \eta} \right) + \beta k^2 \varepsilon u = 0
\]

(1)

where \( \beta = 1 + \eta/\rho \), \( \rho \) is the curvature radius of the ray, and \( k \) is the vacuum wave number. The dielectric permeability \( \varepsilon = \varepsilon_p + \varepsilon_n \) consists of the regular component \( \varepsilon_p \) that is independent of the wave field and the irregular component \( \varepsilon_n \) describing the nonlinear perturbation of the dielectric permeability. Let us find the amplitude of the wave field \( u \) in the form of a localized solution in the vicinity of the ray. The nonlinear perturbation of the dielectric permeability \( \varepsilon_n \) obeys an ordinary differential equation of diffusion type:

\[
H^2 \frac{\partial^2 \varepsilon_n}{\partial \eta^2} - \varepsilon_n + \alpha |u|^2 = 0.
\]

(2)

This equation is valid in the vicinity of the ray at small \( \eta \). Here \( H \) is the scale of thermal diffusion of electrons and \( \alpha \) is the coefficient of the nonlinearity. Note that \( \varepsilon_n = \alpha |u|^2 \) at \( H=0 \).

As usual, we find the solution of the Helmholtz equation (1) in the following form:

\[
u = v \exp(ik\Psi),
\]

where \( \Psi = \int \sqrt{\varepsilon_\sigma} d\xi \), \( v \) is a slowly-varying complex function describing the localized solution, and \( \varepsilon_\sigma \) is the value of the regular component of \( \varepsilon \) on the ray path.

Since we restrict ourselves to the analysis of localized solutions that have maxima \( A_\sigma \) in their domain, it is convenient to normalize the solution to this maximum value and introduce the parameter \( \mu = \sqrt{\alpha A_\sigma^2} \).

If the nonlinearity is weak, the localized solution propagates in an inhomogeneous medium along the nonlinearly perturbed ray path without any broadening despite the fact that the geometrical-optics rays are strongly divergent. A detailed analysis of such solutions is beyond the scope of this paper.
that characterize the degree of nonlinearity of the problem. Then, using the dimensionless coordinates
\( y = k\eta \) and \( x = k\xi \), we write Eqs. (1) and (2) as follows:

\[
\frac{\partial^2 V}{\partial y^2} + 2i \frac{\partial V}{\partial x} + \xi V = 0,
\]

\[
\tilde{H}^2(x) \frac{d^2 \xi}{dy^2} - \xi - |V|^2 = 0,
\]

where \( \tilde{H} = k\eta H, \xi = \frac{\xi}{\mu^2}, V = \frac{v}{A_0}, \) and \( i \) is imaginary unity. The above equation is valid at small \( \mu \) in the
main approximation in the vicinity of the maximum of the electron density, where we put \( \xi = 1 \). In this
case, the ray path is rectilinear, the Helmholtz equation is reduced to the Schrödinger one, while the form
of the diffusion equation remains unchanged. The obtained system of equations (3) allows for describing the
propagation of localized waves in a homogeneous medium with finite heat conductivity of electrons. This is
done by reconstructing the unidirected solution \( V \) in the given region \( D(x, y) \) of the steady-state solution of
the system (3).

We considered a simple physical problem related to the given system of equations. Assume that the
heat conductivity scale varies along the trace \( x \) in the following way:

\[
\tilde{H}(x) = \tilde{H}_{\text{max}} \frac{1 + \cos \left( \frac{\pi (2x/L - 1)}{2} \right)}{1 + \cos \left( \frac{\pi (2x/L - 1)}{2} \right)},
\]

where \( \tilde{L} = k\eta L \) and \( L \) is the characteristic scale of the transverse diffusion coefficient \( H \) over the propagation
direction. It is seen from this formula that the nonlinearity is local at the beginning of the trace, then it
has maximum at \( x = \tilde{L}/2 \), and nullifies again at \( x = \tilde{L}, \tilde{H} \). The ordinary soliton was used as the initial
condition for the nonlinear Schrödinger equation at \( x = 0 \). To solve the system of equations (3) numerically,
it is convenient to choose the integration step much less than unity. Using the special boundary condition,
that eliminates reflection from the upper and lower boundaries of the region \( D \) at \( \eta = \pm 7\pi \), and taking
into account the initial conditions formulated above, we arrive at the well-posed mixed problem. We proved
numerically that the finite-difference problem is stable and converges to the differential one.

The problem described above was solved at various values of the parameters \( \tilde{H}_{\text{max}} \) and \( \tilde{L} \). Introducing
the ratio of these parameters \( B = \tilde{H}_{\text{max}}/\tilde{L} \), we can explain the results of our studies as follows.

If \( B \ll 1 \), the localized solution changes gradually along the direction of propagation. The character-
istic width of this solution increases over the first half of the trace, and decreases gradually over the second
half of the ray path. Thus, the solution at the end of the trace is identical to the initial profile, i.e., we
obtained a soliton. The normalized amplitude \( V_x(y) \) at various \( x \) is shown at Fig. 1. If \( B \) is of the order of
or greater than unity, the solution at the end of the trace differs from the initial one. This means that the
process of propagation is no longer adiabatic. Thus, the diffusion results in significant broadening of the
localized solutions in nonlocal nonlinear media.

If the diffusion at fixed \( x \) (i.e., at fixed \( H \)) is taken into account, the initial problem (Eqs. (1) and (2))
can be reduced to the nonlinear one-dimensional integro-differential Hartree equation

\[
\frac{d^2 V}{dy^2} + V \int_{-\infty}^{+\infty} V^2(t) G(y - t) \, dt = pV,
\]

where the function \( G \), or the fundamental solution, obeys Eq. (3) in which the Dirac delta-function is used
as \( |V|^2, p = \alpha A_0^2 \left[ \frac{\partial \psi}{\partial x} \right]^2 - \xi_p \), and \( \frac{\partial \psi}{\partial x} \) is the partial derivative of the phase at arbitrary fixed \( x \). Then, if
the width of the localized solution is small as compared to the characteristic scale of the regular component
of the dielectric permeability, \( p \) can be approximated by a numerical constant.

To solve the Hartree equation in the considered formulation, one should find the major eigenvalue
and the corresponding eigenfunction. The major eigenvalue determines the phase velocity \( \frac{d\psi}{dx} \) of the localized
solution. We should note that the localized solution is reduced to the classical soliton while the parameter

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