On real Cartan factors

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Received May 22, 1996;
in revised form September 30, 1996

JBW*-triples can be described (modulo W*-algebras, compare [13]) by those of type I. Among these the (complex) Cartan factors are the building blocks. We determine for every complex Cartan factor \( U \) all conjugations of the underlying complex Banach space and hence all real forms (in the sense of [15]) of \( U \), called real Cartan factors. We also give a concrete list of all isomorphy classes of real Cartan factors which generalizes the classification of LOOS [23] to infinite dimensions. Furthermore, we give an explicit description of the full automorphism group as well as the group of all surjective \( \mathbb{R} \)-linear isometries for every non-exceptional real Cartan factor and decide which of the real or complex Cartan factors are isometrically equivalent to each other as real Banach spaces.

1. Introduction

On a complex Banach space \( U \) a conjugation is a conjugate linear isometry \( \tau: U \to U \) with \( \tau^2 = 1_U \) and for every such \( \tau \) the real Banach space \( F := \text{Fix}(\tau) \subset U \) is called a real form of \( U \). Clearly \( \tau \) and \( F \subset U \) determine each other in a unique way. For instance, if \( U = H \) is a complex Hilbert space every orthonormal basis \( (e_i)_{i \in I} \) of \( H \) determines a conjugation by

\[
\tau\left( \sum_{i \in I} c_i e_i \right) = \sum_{i \in I} \overline{c_i} e_i
\]
and it is easily verified that every other conjugation $\sigma$ of $H$ is equivalent to $\tau$ in the following sense: There is a surjective (complex-linear) isometry $g: H \to H$ with $\sigma = g\tau g^{-1}$, or equivalently, with $g(\text{Fix}(\tau)) = \text{Fix}(\sigma)$. In general, a given complex Banach space has many non-equivalent conjugations and also the case occurs that it has no conjugation at all (see [20] for an example).

In this article we study conjugations in a special class of complex Banach spaces, the so called JB*-triples. These form a fairly large class of Banach spaces. For instance every Hilbert space, every Banach space carrying the structure of a C*-algebra or more generally every closed linear subspace $A \subset \mathcal{L}(H, K)$ with $aa^*a \in A$ for every $a \in A$ is in the class, where $\mathcal{L}(H, K)$ is the space of all bounded linear operators from the Hilbert space $H$ into the Hilbert space $K$. These latter operator spaces were introduced by Harris [9] under the name J*-algebras. But there are also JB*-triples which cannot be given as operator spaces, for instance the exceptional JB*-algebras.

Originally [19] the JB*-triples were introduced in connection with the study of bounded symmetric domains in infinite dimensions. These are precisely the complex Banach spaces for which the open unit ball is homogeneous with respect to the group of all biholomorphic automorphisms. A consequence of this is that JB*-triples can also be uniquely characterized by the existence of a certain ternary product $\{xyz\}$, the Jordan triple product. On a C*-algebra for instance this product just is $(xy^*z + zy^*x)/2$. The important fact is that for every JB*-triple the geometry of the Banach space and the algebraic structure given by the triple product determine each other. In particular, on every JB*-triple $U$ the conjugations in the Banach space sense and the triple conjugations (i.e. conjugate linear endomorphisms of period 2 that respect the triple product) are precisely the same. This makes it possible to deal with isometries and conjugations entirely in the algebraic context given by the triple product.

Building blocks for JB*-triples are the Cartan factors. These come in six types, the rectangular operator spaces $\mathcal{L}(H, K)$, spaces of symmetric and of alternating operators, spin factors and two exceptional spaces of dimensions 16 and 27. On every non-exceptional Cartan factor $U$ we determine explicitly all conjugations $\tau$ of $U$ and also the corresponding equivalence classes of them. Calling every real form $F: = \text{Fix}(\tau) \subset U$ a real Cartan factor this means the classification of all real Cartan factors up to isomorphy. This result extends the classification of all real bounded symmetric domains [23] to infinite dimensions. In contrast to the complex case the group $\text{Imt}(F)$ of all surjective linear