PROBABILITY DENSITY OF A STOCHASTIC PROCESS 
AT THE OUTPUT OF A LINEAR SYSTEM

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We propose and analyze a method of direct statistical analysis for linear first-order systems with deterministic parameters. We derive the one- and multi-dimensional (with arbitrary dimension n) probability density of a stochastic process at the output of a linear system.

1. INTRODUCTION

Considerable progress reached in the recent decades in the field of development of numerical methods of transfer and processing of information opens up a number of new possibilities with respect to the filtering, transformation, and amplification of signals. The achievements in the applied field are much ahead of the slowly developing theoretical study of numerous problems related to the nonlinear processing of signals and subsequent filtering of components at the input of an inertialess nonlinear four-terminal circuit [1, 2]. Several approximate methods were developed [2–6] to solve the problem of seeking the statistical characteristics of random processes at the output of linear systems when a non-Gaussian stochastic process is acting at their input. At present, there is no method that allows us in the general case to obtain a multidimensional probability density (PD) with dimension n ≥ 2 at the output of a linear system. The solution of this problem becomes much simpler and more demonstrative and the results become more accurate and reliable, and, which is the most important, it becomes possible to find the multidimensional (with an arbitrary dimension n) PD of the random process at the output of the linear system in the general case if the probability characteristics of the linear systems are determined. As one of such characteristics of the linear systems, we use an n-dimensional PD $q_n(h_0, h_1, \ldots, h_{n-1}; t_0, t_1, \ldots, t_{n-1})$ of instantaneous values of the pulsed characteristic $h(t_0, t)$ of the linear system (or of the significant part of $h(t_0, t)$ if the pulsed characteristic contains a delta function). For the linear systems with deterministic parameters, the above n-dimensional PD is determined in the form

$$q_n(h_0, h_1, \ldots, h_{n-1}; t_0, t_1, \ldots, t_{n-1}) = \prod_{k=0}^{n-1} \delta[h_k - h_r(t_0, t_k)], \quad (1)$$

where $\delta(x)$ is the delta function, $t$ is the current time, $t_0$ is the time at which the external action is applied, $k, n$ are the summation subscripts, and "r" is subscript indicating the “regularity” (determinacy) of the function $h(t_0, t)$; \{h_0, h_1, \ldots, h_{n-1}\} is the set of values of the pulsed characteristic $h(t_0, t)$ corresponding to the times $t_0, t_1, \ldots, t_{n-1}$.

Here and below, for the sake of convenience, we use PDs with dimensions n (see Eq. (1)) and (n + 1):

$$q_{n+1}(h_0, h_1, \ldots, h_n; t_0, t_1, \ldots, t_n) = \prod_{k=0}^{n} \delta[h_k - h_r(t_0, t_k)],$$

$$q_n(h_1, h_2, \ldots, h_n; t_1, t_2, \ldots, t_n) = \prod_{k=1}^{n} \delta[h_k - h_r(t_0, t_k)], \quad (2)$$


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and discuss only one-dimensional linear systems, which does not diminish the generality of the results obtained [7]. We confine our discussion to first-order linear systems because it is necessary to consider the essence of the proposed method of statistical analysis of linear systems in a simpler and more demonstrative form. Its generalization to higher-order linear systems and multidimensional linear systems with random parameters involves no theoretical difficulties.

In this paper we use the proposed method to find the one-dimensional and multi-dimensional PDs of instantaneous values of the stochastic process \( x(t_0, t) \) at the output of a one-dimensional first-order linear system with deterministic parameters in the case of the given PD \( w_{n+1}(\xi_0, \xi_1, \ldots, \xi_n; t_0, t_1, \ldots, t_n) \) with dimension \((n+1)\) of the stochastic process \( \xi(t) \) at its input and zero noise \( z(t) \) of the linear system.

In the class of linear systems under study, we separate two subclasses discussed below.

2. LINEAR SYSTEMS WHOSE PULSED CHARACTERISTICS CONTAIN NO DELTA FUNCTION

Many such linear systems are described by linear differential equations. A specific group is formed by Gaussian and \( \Pi \)-shaped (ideal) linear filters (the so-called “limit” filters) for which it is impossible to write equations like

\[
a_1(t) \frac{dx_1(t_0, t)}{dt} + a_0(t) \cdot x_1(t_0, t) = \xi(t)
\]

with the initial condition

\[
x_1(t_0, t) \bigg|_{t=t_0} = x_1,
\]

where \( \xi(t) \) and \( x_1(t_0, t) \) are the input action and the linear-system response, respectively; \( a_1(t) \) and \( a_0(t) \) are variable coefficients of Eq. (3) that are “slow” compared with \( x_1(t_0, t) \), i.e., \( \Delta f_x \gg \Delta f_{a_1}; \Delta f_x \) and \( \Delta f_{a_1} \) are the widths of the energy spectra of the functions \( x_1(t_0, t) \) and \( a_1(t) \), respectively; \( i = 0, 1 \).

The pulsed characteristics of such linear systems are expressed in terms of the coefficients of Eq. (3):

\[
h(t_0, t) = \frac{1}{a_1(t)} \exp \left[ - \int_{t_0}^{t} \frac{a_0(\tau)}{a_1(\tau)} d\tau \right].
\]

Free and induced motion of the linear system. The response \( x_1(t_0, t) \) of the linear system includes the free motion \( x_{fr}(t_0, t) \) due to nonzero initial conditions (4) and the induced motion \( x_{in}(t_0, t) \) corresponding to the zero initial conditions [4, 7]

\[
x_1(t_0, t) = x_{fr}(t_0, t) + x_{in}(t_0, t) = a_1(t_0) \cdot x_1 \cdot h(t_0, t) + \int_{t_0}^{t} \xi(\tau) h(t_0, t - \tau) d\tau.
\]

One-dimensional and multi-dimensional probability density of the free motion of a linear system. For the subclass of linear systems considered in this paper, the free motion is the regular function (see Eqs. (5) and (6)) \( x_{fr}(t_0, t) = a_1(t_0) \cdot x_1 \cdot h(t_0, t) \). In accordance with Eq. (1), the PD of the arbitrary dimension \((n+1)\) of the free motion \( x_{fr}(t_0, t) \) in the interval \( \Delta t = t - t_0 \) has the form

\[
q_{n+1}(x_{fr0}, x_{fr1}, \ldots, x_{frn}; t_0, t_1, \ldots, t_n) = \prod_{k=0}^{n} \delta [x_{frk} - x_{fr}(t_0, t_k)] = \prod_{k=0}^{n} \delta [x_{frk} - a_1(t_0) x_1 h(t_0, t_k)].
\]

Eq. (7), for the arbitrary time \( t_i \), from the interval \( \Delta t \), we obtain the one-dimensional PD

\[
q_1(x_{fri}; t_0, t_i) = \delta [x_{fri} - a_1(t_0) x_1 h(t_0, t_i)].
\]

One-dimensional probability density of the estimate of the induced motion of a linear