A Boundary Value Problem for Poisson's Two-Media Equation in $L_p^2$-Spaces

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ABSTRACT. In the paper we study a binding boundary value problem for two media for Poisson's equation

$$\mu \Delta u = f(x),$$

with solutions in the class $L_p^2(\mathbb{R}_+^3), \quad 1 < p < \infty,$ with the corresponding seminorm, where

$$\mathbb{R}_+^3 = \{ x \mid x' = (x_1, x_2) \in \mathbb{R}^2, \quad x_3 \geq 0 \}, \quad \mu = \begin{cases} \mu_+, & x_3 > 0, \\ \mu_-, & x_3 < 0. \end{cases}$$

It is proved that the solution exists for all $f(x) \in L_p^2,$ and a priori estimates of the solution are obtained with the help of multiplicators in the space $L_p^2(\mathbb{R}_+^3).$ An explicit solution of the problem for all $f(x) \in \dot{C}^1(\mathbb{R}^3)$ is obtained. The kernel of the operator generated by the problem is constructed (in explicit form) as a polynomial of the first degree.

KEY WORDS: Poisson's equation, binding boundary value problem, Laplace operator, kernel of a differential operator, multiplicator, Hardy's inequality, distribution theory, Lizorkin's theorem, Fubini's theorem.

With the view of studying binding problems for the Stokes and Navier-Stokes systems in the future, including problems with an unknown boundary [1, 2], in this paper we study a binding problem for Poisson's two-media equation

$$\mu \Delta u = f, \quad x \in \mathbb{R}_+^3 \cup \mathbb{R}_-^3,$$  \hspace{1cm} (1)

where

$$\mathbb{R}_+^3 = \{ x \mid x' = (x_1, x_2) \in \mathbb{R}^2, \quad x_3 \geq 0 \}, \quad \mu = \begin{cases} \mu_+, & x_3 > 0, \\ \mu_-, & x_3 < 0, \end{cases}$$

and $\mu_+, \mu_-$ are positive constants. The binding conditions at the interface between the two media, i.e., in the hyperplane $x_3 = 0,$ are of the form

$$\left. u \right|_{x_3=+0} = u \left|_{x_3=-0}, \right.$$  \hspace{1cm} (2)

$$\left. \frac{\partial u}{\partial n} \right|_{x_3=+0} = \mu_+ \left. \frac{\partial u}{\partial n} \right|_{x_3=-0}.$$  \hspace{1cm} (3)

Problem (1)-(3) models a stationary temperature distribution with different thermal conductivities for the two media. The solution is studied in the space $u(x) \in L_p^2(\mathbb{R}_+^3 \cup \mathbb{R}_-^3)$ (for brevity $L_p^2(\mathbb{R}_+^3)$) with seminorm

$$\| u \|_{L_p^2(\mathbb{R}_+^3)} = \sum_{|\alpha|=2} \| D^\alpha u \|_{L_p(\mathbb{R}^3)} \quad 1 < p < \infty, \quad f(x) \in L_p(\mathbb{R}^3).$$

Since the region in which we consider the solution is unbounded, the condition at infinity is sometimes used:

$$\lim_{|z| \to \infty} u(x) = 0.$$  \hspace{1cm} (4)

In the present paper, the solution of problem (1)-(4) for $f \in \dot{C}^\infty(\mathbb{R}^3)$ is constructed in explicit form, the following estimate of the second derivatives of the solution is obtained in the $L_p^2(\mathbb{R}_+^3)$ seminorm:

$$\sum_{|\alpha|=2} \| D_+^\alpha u \|_{L_p(\mathbb{R}_+^3)} + \sum_{|\alpha|=2} \| D_-^\alpha u \|_{L_p(\mathbb{R}_-^3)} \leq M \| f \|_{L_p(\mathbb{R}^3)}.$$
with constant $M > 0$ independent of $u(x)$, and the existence of the solution of problem (1)-(3) is proved for all $f(x) \in L_p(\mathbb{R}^3)$. It is also proved that the dimension of the kernel of the operator generated by problem (1)-(3), with domain $L^2_\mu(\mathbb{R}^3)$ and the range of values $L_p(\mathbb{R}^3)$ is equal to 4. For the proof of the results, the theory of multiplicators of Fourier transforms [3], Hardy's inequalities [4, 5], and the theory of distributions [6] are used.

§1. Construction of an explicit solution

We shall construct a solution of problem (1)-(4) for $f(x) \in \hat{C}^\infty(\mathbb{R}^3)$. The following theorem is valid.

**Theorem 1.** If $f(x) \in \hat{C}^\infty(\mathbb{R}^3)$ and condition (4) is satisfied, then the solution of problem (1)-(3) is of the form

$$u(x', x_3) = F_{\xi' \to \xi}^{-1}(\hat{u}(\xi', x_3)),$$

where

$$\hat{u}(\xi', x_3) = \frac{e^{-|\xi'|x_3}}{(\mu_- - \mu_+)|\xi'|} \int_{-\infty}^{+\infty} \hat{f}(\xi', y_3)e^{i|\xi'|y_3} dy_3$$

$$+ \frac{e^{-i|\xi'|x_3}(\mu_- + \mu_+)}{2\mu_+|\mu_- + \mu_+|^2} \int_{-\infty}^{+\infty} \hat{f}(\xi', y_3)e^{-i|\xi'|y_3} dy_3$$

$$- \frac{e^{-i|\xi'|x_3}}{2\mu_-|\xi'|} \int_{0}^{x_3} \hat{f}(\xi', y_3)e^{i|\xi'|y_3} dy_3, \quad x_3 > 0,$$

$$\hat{u}(\xi', x_3) = \frac{e^{i|\xi'|x_3}(\mu_- + \mu_+)}{2\mu_-|\mu_- + \mu_+|^2} \int_{0}^{-\infty} \hat{f}(\xi', y_3)e^{i|\xi'|y_3} dy_3$$

$$+ \frac{e^{i|\xi'|x_3}}{2\mu_-|\xi'|} \int_{0}^{x_3} \hat{f}(\xi', y_3)e^{-i|\xi'|y_3} dy_3, \quad x_3 < 0.$$

**Proof.** Applying the Fourier transform over $x' = (x_1, x_2)$ to (1), we obtain

$$\frac{\partial^2 \hat{u}}{\partial x_3^2} - |\xi'|^2 \hat{u} = \frac{\hat{f}(\xi', x_3)}{\mu}.$$

The general solution of the homogeneous equation corresponding to (7) is of the form

$$\hat{u}_0(\xi', x_3) = a(\xi')e^{-|\xi'|x_3} + b(\xi')e^{i|\xi'|x_3}.$$

Let us obtain a particular solution of Eq. (7). Consider the Cauchy problem

$$\frac{\partial^2 \hat{u}}{\partial x_3^2} - |\xi'|^2 \hat{u} = \frac{\hat{f}(\xi', x_3)}{\mu}, \quad \hat{u}(\xi', 0) = \hat{u}'_{\xi_3}(\xi', 0) = 0.$$

Its solution is of the form

$$\hat{u}(\xi', x_3) = \int_{0}^{x_3} \varphi(\xi', x_3 - y_3)\frac{\hat{f}(\xi', y_3)}{\mu} dy_3,$$

where $\varphi(\xi', x_3)$ is the solution of the problem

$$\varphi''_{x_3} - |\xi'|^2 \varphi = 0, \quad \varphi(\xi', 0) = 0, \quad \varphi'_{x_3}(\xi', 0) = 1,$$

$\varphi''_{x_3} - |\xi'|^2 \varphi = 0, \quad \varphi(\xi', 0) = 0, \quad \varphi'_{x_3}(\xi', 0) = 1$.