A Boundary Value Problem for Poisson's Two-Media Equation in $L^p_\pm$-Spaces

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ABSTRACT. In the paper we study a binding boundary value problem for two media for Poisson's equation

$$\mu \Delta u = f(x)$$

with solutions in the class $L^p_\pm(\mathbb{R}^3_\pm)$, $1 < p < \infty$, with the corresponding seminorm, where

$$\mathbb{R}^3_\pm = \{ x \mid x' = (x_1, x_2) \in \mathbb{R}^2, \ x_3 \geq 0 \}, \quad \mu = \begin{cases} \mu_+, \ x_3 > 0, \\ \mu_-, \ x_3 < 0. \end{cases}$$

It is proved that the solution exists for all $f(x) \in L^p_\pm$, and a priori estimates of the solution are obtained with the help of multiplicators in the space $L^2_\pm(\mathbb{R}^3_\pm)$. An explicit solution of the problem for all $f(x) \in \mathcal{C}(\mathbb{R}^3)$ is obtained. The kernel of the operator generated by the problem is constructed (in explicit form) as a polynomial of the first degree.

KEY WORDS: Poisson's equation, binding boundary value problem, Laplace operator, kernel of a differential operator, multiplicator, Hardy's inequality, distribution theory, Lizorkin's theorem, Fubini's theorem.

With the view of studying binding problems for the Stokes and Navier–Stokes systems in the future, including problems with an unknown boundary [1, 2], in this paper we study a binding problem for Poisson's two-media equation

$$\mu \Delta u = f, \quad x \in \mathbb{R}^3_+ \cup \mathbb{R}^3_-, \quad (1)$$

where

$$\mathbb{R}^3_\pm = \{ x \mid x' = (x_1, x_2) \in \mathbb{R}^2, \ x_3 \geq 0 \}, \quad \mu = \begin{cases} \mu_+, \ x_3 > 0, \\ \mu_-, \ x_3 < 0, \end{cases}$$

and $\mu_+, \mu_-$ are positive constants. The binding conditions at the interface between the two media, i.e., in the hyperplane $x_3 = 0$, are of the form

$$u|_{x_3=0} = u|_{x_3=-0}, \quad (2)$$

$$\mu_+ \frac{\partial u}{\partial n} \bigg|_{x_3=0} = \mu_- \frac{\partial u}{\partial n} \bigg|_{x_3=-0}. \quad (3)$$

Problem (1)–(3) models a stationary temperature distribution with different thermal conductivities for the two media. The solution is studied in the space $u(x) \in L^p_\pm(\mathbb{R}^3_+ \cup \mathbb{R}^3_-)$ (for brevity $L^p_\pm(\mathbb{R}^3_\pm)$) with seminorm

$$\|u\|_{L^p_\pm(\mathbb{R}^3_\pm)} = \sum_{|\alpha|=2} \|D^\alpha u\|_{L^p(\mathbb{R}^3_\pm)}, \quad 1 < p < \infty, \quad f(x) \in L^p(\mathbb{R}^3).$$

Since the region in which we consider the solution is unbounded, the condition at infinity is sometimes used:

$$\lim_{|x| \to \infty} u(x) = 0. \quad (4)$$

In the present paper, the solution of problem (1)–(4) for $f \in \mathcal{C}_\infty(\mathbb{R}^3)$ is constructed in explicit form, the following estimate of the second derivatives of the solution is obtained in the $L^p_\pm(\mathbb{R}^3_\pm)$ seminorm:

$$\sum_{|\alpha|=2} \|D^\alpha_\pm u\|_{L^p(\mathbb{R}^3_\pm)} + \sum_{|\alpha|=2} \|D^\alpha u\|_{L^p(\mathbb{R}^3_-)} \leq M\|f\|_{L^p(\mathbb{R}^3)}$$
with constant $M > 0$ independent of $u(x)$, and the existence of the solution of problem (1)-(3) is proved for all $f(x) \in L_p(S)$. It is also proved that the dimension of the kernel of the operator generated by problem (1)-(3), with domain $L^2_P(S^1)$ and the range of values $L^p(S^3)$ is equal to 4. For the proof of the results, the theory of multiplicators of Fourier transforms [3], Hardy’s inequalities [4, 5], and the theory of distributions [6] are used.

§1. Construction of an explicit solution

We shall construct a solution of problem (1)-(4) for $f(x) \in C^\infty(S^3)$. The following theorem is valid.

**Theorem 1.** If $f(x) \in C^\infty(S^3)$ and condition (4) is satisfied, then the solution of problem (1)-(3) is of the form $u(x', x_3) = F_{x_3}^{-1} [\hat{u}(x', x_3)]$, where

$$
\hat{u}(x', x_3) = \frac{e^{-|x'| x_3}}{(\mu_- - \mu_+)|x'|} \int_0^{-\infty} f(x', y_3) e^{i|x'| y_3} dy_3
$$

$$
+ \frac{e^{-|x'| x_3}(\mu_- + \mu_+)}{2(\mu_- - \mu_+)|x'|} \int_0^{+\infty} f(x', y_3) e^{-|x'| y_3} dy_3 - \frac{e^{i|x'| x_3}}{2(\mu_- + |x'|)} \int_{x_3}^{+\infty} f(x', y_3) e^{-i|x'| y_3} dy_3
$$

$$
- \frac{e^{-i|x'| x_3}}{2(\mu_+ + |x'|)} \int_0^{x_3} f(x', y_3) e^{i|x'| y_3} dy_3, \quad x_3 > 0,
$$

$$
\hat{u}(x', x_3) = \frac{e^{i|x'| x_3}(\mu_- + \mu_+)}{2(\mu_- - \mu_+)|x'|} \int_0^{-\infty} f(x', y_3) e^{i|x'| y_3} dy_3
$$

$$
+ \frac{e^{-|x'| x_3}}{(\mu_- + \mu_+)|x'|} \int_0^{+\infty} f(x', y_3) e^{-|x'| y_3} dy_3 - \frac{e^{-i|x'| x_3}}{2(\mu_- + |x'|)} \int_{x_3}^0 f(x', y_3) e^{-i|x'| y_3} dy_3
$$

$$
+ \frac{e^{i|x'| x_3}}{2(\mu_+ + |x'|)} \int_0^{x_3} f(x', y_3) e^{i|x'| y_3} dy_3, \quad x_3 < 0.
$$

**Proof.** Applying the Fourier transform over $x' = (x_1, x_2)$ to (1), we obtain

$$
\frac{\partial^2 \hat{u}}{\partial x_3^2} - |x'|^2 \hat{u} = \frac{\hat{f}(x', x_3)}{\mu}.
$$

The general solution of the homogeneous equation corresponding to (7) is of the form

$$
\hat{u}_0(x', x_3) = a(x')e^{-|x'| x_3} + b(x')e^{i|x'| x_3}.
$$

Let us obtain a particular solution of Eq. (7). Consider the Cauchy problem

$$
\frac{\partial^2 \hat{u}}{\partial x_3^2} - |x'|^2 \hat{u} = \frac{\hat{f}(x', x_3)}{\mu}, \quad \hat{u}(x', 0) = \hat{u}_{x_3}(x', 0) = 0.
$$

Its solution is of the form

$$
\hat{u}(x', x_3) = \int_0^{x_3} \varphi(x', x_3 - y_3) \frac{\hat{f}(x', y_3)}{\mu} dy_3,
$$

where $\varphi(x', x_3)$ is the solution of the problem

$$
\varphi'' - |x'|^2 \varphi = 0, \quad \varphi(x', 0) = 0, \quad \varphi_{x_3}(x', 0) = 1.
$$

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1 At the authors’ request, several minor corrections were incorporated in the text; here, for example, 4 was misprinted as 1 in the Russian version.