On Integral Convolution Operators

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UDC 517.948.5

Abstract. We study integral convolutions defined on functions of n variables in symmetric spaces and obtain new additive estimates for the mean value of the nonincreasing permutation \((K*f)^{**}(t)\) of the absolute value of the integral convolution on the interval \([0, t]\) for any \(t > 0\). As an example of application of the estimates obtained, we prove the boundedness of the integral convolution operator acting from the intersection of symmetric spaces into the Marcinkiewicz space.

Key words: integral convolution, symmetric space, dual space, measurable function, O'Neill inequality, Hölder inequality.

The paper deals with the integral convolutions

\[(K*f)(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy \quad (1)\]

defined for functions \(K(x)\) and \(f(x)\) measurable on the n-dimensional Euclidean space \(\mathbb{R}^n\).

By \(E(\mathbb{R}^n)\) we denote the symmetric space of functions defined on the n-dimensional Euclidean space \(\mathbb{R}^n\) \((\varphi_E(t)\) is the fundamental function of \(E(\mathbb{R}^n)\)) and by \(E'(\mathbb{R}^n)\) the dual space of \(E(\mathbb{R}^n)\).

Let \(g^*(t)\) be a nonincreasing permutation of \(|g(x)|\), where \(g(x)\) is a function measurable on \(\mathbb{R}^n\), let \(\chi_{[a,b]}(t)\) be the characteristic function of the interval \([a, b]\), and let

\[g^{**}(z) = z^{-1} \int_0^z g^*(t)\,dt.\]

By \(\Phi\) we denote the set of continuous positive functions \(\varphi(t)\) nondecreasing on the semiaxis \((0, \infty)\) and satisfying the conditions \(\varphi(0+) = 0\) and \(\varphi(\infty) = \infty\). For any function \(\varphi(z) \in \Phi\) and \(s > 0\) we define the dilatation function

\[M_{\varphi}(s) = \sup_{z > 0} \frac{\varphi(sz)}{\varphi(z)}.

Theorem 1. Suppose that a nonincreasing permutation \(K^*(t)\) of the kernel of the operator \((1)\) is integrable on the interval \([0, 1]\) and \(K^*(t)\chi_{[1, \infty]}(t)\) belongs to the space \(E(\mathbb{R})\). Then

\[(K^* f)^{**}(t) \leq 2 \left( \int_0^{\delta(t)} K^*(z)\,dz \cdot f^{**}(t) + \|K^* \chi_{[\delta(t), \infty]}\|_{E'(\mathbb{R})} \cdot \|f\|_{E(\mathbb{R}^n)} \right) \quad (2)\]

for any functions \(f(x) \in E(\mathbb{R}^n)\) and \(\delta(z) \in \Phi\) and for any \(t > 0\).

Proof. Suppose that a function \(\delta(z)\) belongs to \(\Phi\) and \(t > 0\). For an arbitrary function \(f(x) \in E(\mathbb{R}^n)\) and the kernel of the integral convolution operator satisfying the assumptions of the theorem, from the O"{N}e"{n} [1] inequality with \(\delta(t) < t\) we obtain

\[(K^* f)^{**}(t) \leq t K^{**}(t)f^{**}(t) + \int_{1}^{\infty} K^*(z)f^*(z)\,dz \leq \int_0^{\delta(t)} K^*(z)\,dz \cdot f^{**}(t) + \int_{\delta(t)}^{t} K^*(z)\,dz \cdot f^*(t) + \int_{t}^{\infty} K^*(z)f^*(z)\,dz. \quad (3)\]
By using the assumptions imposed on the kernel $K(x)$ in the theorem and a Hölder type inequality for symmetric spaces [2], we arrive at

$$
\int_{\delta(t)}^{t} K^*(z) \, dz \cdot f^{**}(t) \leq \left\| K^* \chi_{[\delta(t), \infty)} \right\|_{E'(\mathbb{R}^n)} \cdot \varphi_{E}(t) \varphi_{E'}(t) \cdot \frac{1}{t} \left\| f^* \chi_{[0, t]} \right\|_{E(\mathbb{R}^n)}.
$$

$$
\int_{t}^{\infty} K^*(z) f^*(z) \, dz \leq \left\| K^* \chi_{[t, \infty)} \right\|_{E'(\mathbb{R}^n)} \cdot \left\| f^* \chi_{[t, \infty)} \right\|_{E(\mathbb{R}^n)}.
$$

By substituting these inequalities into (3) and taking into account the inequality $\varphi_{E}(t) \varphi_{E'}(t) \leq t$, we obtain

$$
(K^* f)^{**}(t) \leq \int_{0}^{\delta(t)} K^*(z) \, dz \cdot f^{**}(t) + 2 \left\| K^* \chi_{[\delta(t), \infty)} \right\|_{E'(\mathbb{R}^n)} \cdot \left\| f \right\|_{E(\mathbb{R}^n)}.
$$

For $\delta(t) \geq t$ the O'Neil inequality implies

$$
(K^* f)^{**}(t) \leq t K^{**}(t) f^{**}(t) + \int_{0}^{\delta(t)} K^*(z) f^*(z) \, dz + \int_{\delta(t)}^{\infty} K^*(z) f^*(z) \, dz
$$

$$
\leq 2 \int_{0}^{\delta(t)} K^*(z) \, dz \cdot f^{**}(t) + \left\| K^* \chi_{[\delta(t), \infty)} \right\|_{E'(\mathbb{R}^n)} \cdot \left\| f \right\|_{E(\mathbb{R}^n)}.
$$

The theorem is thereby complete. □

**Theorem 2.** Suppose that concave functions $\varphi(z)$ and $\psi(z)$ from the set $\Phi$ satisfy the condition

$$
\int_{0}^{1} \frac{dz}{\varphi(z)} + \int_{1}^{\infty} \frac{d\psi(z)}{\varphi(z)} < \infty,
$$

and $E(\mathbb{R}^n)$ is the symmetric space with fundamental function $\varphi_{E}(t) = t/\varphi(t)$. If the nonincreasing permutation of the kernel $K(x)$ of the operator (1) satisfies the condition

$$
\sup_{t>0} K^*(t) \varphi(t) < \infty, \quad (4)
$$

then

$$
(K^* f)^{**}(t) \leq C \left( \int_{0}^{\delta(t)} \frac{dz}{\varphi(z)} + \int_{\delta(t)}^{\infty} \frac{d\psi(z)}{\varphi(z)} \right) \cdot \left\| f \right\|_{E(\mathbb{R}^n)}
$$

for any functions $f(x) \in E(\mathbb{R}^n)$ and $\delta(z) \in \Phi$ and an arbitrary $t > 0$. Here the constant $C$ is independent of $f(x)$, $\delta(t)$, and $t$.

**Proof.** It follows from the assumptions of Theorem 2 that the assumptions of Theorem 1 are satisfied. By using inequality (4), from Theorem 1 we obtain

$$
(K^* f)^{**}(t) \leq C \left( \int_{0}^{\delta(t)} \frac{dz}{\varphi(z)} + \int_{0}^{t} f^*(z) \, dz + \left\| \chi_{[\delta(t), \infty)} \varphi(z) \right\|_{E'(\mathbb{R}^n)} \cdot \left\| f \right\|_{E(\mathbb{R}^n)} \right).
$$

Next, we estimate the integral with the help of a Hölder type inequality for symmetric spaces as follows:

$$
\int_{0}^{t} f^*(z) \, dz \leq \varphi_{E'}(t) \left\| f \right\|_{E(\mathbb{R}^n)} = \psi(t) \left\| f \right\|_{E(\mathbb{R}^n)}.
$$

From the embedding of the Lorentz space into the symmetric space with the same fundamental function [2], we obtain

$$
\left\| \chi_{[\delta(t), \infty)} \varphi(z) \right\|_{E'(\mathbb{R}^n)} \leq \int_{\delta(t)}^{\infty} \frac{d\psi(z)}{\varphi(z)}.
$$

By substituting these inequalities into inequality (5), we complete the proof of Theorem 2. □