Varieties of Solvable Index-Two Alternative Algebras 
Over a Field of Characteristic Three

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ABSTRACT. The subvarieties of the variety $\text{Alt}_2$ of solvable index-two alternative algebras over an arbitrary field of characteristic 3 are studied. The main types of such varieties are singled out in the language of identities, and inclusions between these types are established. The main result is the following.

**Theorem.** The topological rank of the variety $\text{Alt}_2$ of solvable index-two alternative algebras over an arbitrary field of characteristic 3 is equal to five.

KEY WORDS: variety of algebras, alternative algebra, identity in an algebra, topological rank.

The objective of the present note is to prove the following theorem.

**Theorem.** The topological rank of the variety $\text{Alt}_2$ of solvable index-two alternative algebras over an arbitrary field of characteristic 3 is equal to five.

As is known [1], the variety $\text{Alt}_2$ over a field whose characteristic differs from 2 and 3 has two nonnilpotent subvarieties. In particular, this implies that the topological rank of the variety $\text{Alt}_2$ is two. The study of the proofs shows that these results hold in the case of a field of characteristic two as well.

Recall that the Medvedev Theorem [2] on the Specht property of the variety of algebras satisfying a two-term identity yields the Specht property of the varieties of solvable index-two algebras that are nearly associative and, in particular, this holds for the variety $\text{Alt}_2$. The structure of the lattice of subvarieties of the variety $\text{Alt}_2$ over a field of characteristic zero in which the equation $t^2 + t + 1 = 0$ is solvable is described by Il'tyakov [3].

The present paper consists of three sections. In §1, the main definitions are given and the preliminary results are proved. In §2, the main types of the subvarieties of $\text{Alt}_2$ are singled out and an upper estimate for the topological rank is proved. The last section, §3, is devoted to the construction of examples needed in order to give a lower estimate for the topological rank.

§1. Preliminary results

1. Space of multilinear polynomials. In what follows, unless otherwise stated, the term “algebra” means a solvable index-two alternative algebra over an arbitrary field $\Phi$ of characteristic 3.

Note that in a solvable index-two alternative algebra, any monomial of degree at least three with respect to some variable is zero, and hence any variety of solvable index-two alternative algebras is defined by homogeneous identities of degree at most two with respect to each of the variables. Therefore, such a variety is homogeneous [4]. Since the characteristic of the field is three, it follows that any variety of solvable index-two alternative algebras can be defined by some system of multilinear identities (polynomials); moreover, we tacitly assume that each identity of degree $n$ depends on variables in the set $X_n = \{x_1, x_2, \ldots, x_n\}$.

Let $M$ be an arbitrary variety of solvable index-two alternative algebras, and let $A = F_M[X]$ be a free $M$-algebra with the set $X = \{x_1, x_2, \ldots\}$ of free generators.

**Definition** [1]. By tame words of the algebra $A$ in the variables belonging to the set $X_n$ ($n \geq 4$) we mean the following multilinear monomials:

a) $(x_1 x_j)R(k_1) \cdots R(k_{n-2})$, $k_1 < k_2 < \cdots < k_{n-2}$,

b) $(x_j x_1)R(k_1) \cdots R(k_{n-2})$, $k_1 < k_2 < \cdots < k_{n-2}$,

c) $(x_2 x_3)R(1) \cdots R(n)$;
where $R(k)$ is the operator of right multiplication by the element $x_k$.

By a regular word of the algebra $A$ in the variables belonging to $X_n$ we mean the word
d) $(x_3(x_1x_2))R(4)\cdots R(n)$.

**Lemma 1** [1]. The space $P_n(A)$ of multilinear monomials over $X_n$ is linearly spanned by the tame and regular words of the algebra $A$.

Introduce the following auxiliary functions that are skew-symmetric with respect to $y, z$:

$$
\{x, y, z\} = (x \circ y)z - (x \circ z)y - (x, y, z), \quad [x, y, z] = [x, y]z - [x, z]y,
$$

where $x \circ y = xy + yx$ and $[x, y] = xy - yx$ (the Jordan product and the commutator, respectively).

We need a modification of the system of tame and regular words that also spans the space $P_n(A)$.

**Lemma 2.** The space $P_n(A)$ $(n \geq 4)$ is spanned by the elements of the following form, which are said to be basis words in what follows:

1) $(x_1 \circ x_1)\rho_i$, $i \geq 4$,
2) $[x_1, x_1]\rho_i$, $i \geq 4$,
3) $(x_1 \circ x_2)\rho_2$,
4) $[x_1, x_2]\rho_2$,
5) $\{x_1, x_2, x_3\}\eta$,
6) $[x_1, x_2, x_3]\eta$,
7) $(x_1, x_2, x_3)\eta$,
8) $[x_1, [x_2, x_3]]\eta$,

where $\rho_i = R(2)\cdots R(i)\cdots R(n)$ is the operator word in which the operator $R(i)$ is absent, and $\eta = R(4)R(5)\cdots R(n)$.

**Proof.** The definition of the Jordan product and the commutator over a field of characteristic 3 yields the relation $xy = 2x \circ y + 2[x, y]$, and hence, for $j \neq 3$, the tame words of types a) and b) are representable as linear combinations of words of types 1) and 2). Denote by $V$ the space generated by the elements of types 3)-8) over $X_4$; for polynomials $x$ and $y$ over $X_3$, we write $x \equiv y$ if $(x - y)R(4) \in V$.

It is clear that

$$
(x_2x_1)x_3 \equiv (x_1x_2)x_3 \equiv 0. \quad (1)
$$

It follows from the definition of the functions $\{x_1, x_2, x_3\}$ and $[x_1, x_2, x_3]$ that

$$(x_1 \circ x_2)x_2 = (x_1 \circ x_2)x_3 - \{x_1, x_2, x_3\} - (x_1, x_2, x_3), \quad [x_1, x_3]x_2 = [x_1, x_2]x_3 - [x_1, x_2, x_3],$$

that is,

$$(x_3x_1)x_2 \equiv (x_1x_3)x_2 \equiv 0. \quad (2)$$

Thus, it follows from the congruences (1) and (2) and from the remark at the very beginning of the proof that all tame words of types a) and b) are representable as linear combinations of basis words. Hence, by Lemma 1, it suffices to verify the congruences $v := (x_2x_3)x_1 \equiv 0$ and $w := x_3(x_1x_2) \equiv 0$.

Applying the congruences (1) and (2) and the fact that the associator is skew-symmetric with respect to all its variables, we obtain

$$w = x_3(x_1x_2) = (x_3x_1)x_2 - (x_3, x_1, x_2) = (x_3x_1)x_2 - (x_1, x_2, x_3) \equiv 0;$$

similarly,

$$x_1(x_2x_3) = (x_1x_2)x_3 - (x_1, x_2, x_3) \equiv 0, \quad (3)
$$

$$x_1(x_3x_2) = (x_1x_3)x_2 - (x_1, x_3, x_2) \equiv 0. \quad (4)$$