BRIEF COMMUNICATIONS

Canonical Form of the Fourth Degree Homogeneous Part in a Normal Equation of a Real Hypersurface in $\mathbb{C}^3$

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Let $M \subset \mathbb{C}^3$ be a real analytic hypersurface with a nondegenerate Levi form. According to [1], there exist coordinates $z = (z_1, z_2), w = u + iv$ (where $u, v$ are respectively the real and the imaginary part of the variable $w$) such that the equation of $M$ takes the form

$$v = \langle z, \bar{z} \rangle + \sum_{k \geq 2, l \geq 2, m \geq 0} F_{klm}(z, \bar{z}, u),$$

where $\langle z, \bar{z} \rangle$ is the Hermitian form and $F_{klm}(z, \bar{z}, u)$ is a polynomial of degrees $k, l, m$ in $z, \bar{z}, u$ respectively. This means that the summand $F_{220}$ is the one of minimal order that can be distinct in the normalized equations of two surfaces with a given Levi form.

Below we study the forms to which one can reduce a nonzero polynomial $F_{220}$ by means of holomorphic transformations preserving the origin and the normal form of the equation of $M$. Such questions are useful, for example, for the local classification of homogeneous manifolds in terms of their normal equations.

We distinguish between the following two cases depending on the signature of the Levi form:

a) $\langle z, z \rangle_+ = |z_1|^2 + |z_2|^2$, the strictly pseudoconvex case;
b) $\langle z, z \rangle_- = \text{Im} z_1 \bar{z}_2$, the case of an indefinite form.

The polynomial $F_{220}$ possesses the "harmonicity" property $\Delta_\pm F_{22} = 0$, where

$$\Delta_+ = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}$$

corresponds to the strictly pseudoconvex case, and

$$\Delta_- = \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} - \frac{\partial^2}{\partial z_2 \partial \bar{z}_1}$$

corresponds to the indefinite Levi form.

We use the same notation $\mathfrak{H}_{22}$ for the space of "harmonic" polynomials in both cases. We prove below, by means of a unified approach, the following two statements.

**Theorem 1.** For the indefinite Levi form a nonzero polynomial $F_{220}(z, \bar{z})$ from the normalized equation (1) can be reduced to one of the following forms:

1) $\sigma_2 + a \sigma_0 + \sigma_{-2}, \ a \geq 2$;
2) $\sigma_1 + a \sigma_0 + \sigma_{-1}, \ a \geq 0$;
3) $\sigma_2 + \sigma_0$;
4) $\sigma_2$;
5) $\sigma_1$;
6) $\sigma_0$.

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\[
\sigma_{-2} = |z_2|^4, \quad \sigma_{-1} = \text{Re}(z_1 \bar{z}_2)|z_2|^2, \quad \sigma_0 = \frac{2}{3}|z_1|^2|z_2|^2 + \frac{1}{3}\text{Re} \ z_1^2 \bar{z}_2^2, \quad \sigma_2 = |z_1|^4, \quad \sigma_1 = \text{Re}(z_1 \bar{z}_2)|z_1|^2.
\]

**Theorem 2.** In the strictly pseudoconvex case a nonzero polynomial \( F_{220}(z, \bar{z}) \) can be reduced to one of the following forms:

1) \( \pm(\eta_{-2} + a\eta_{-1} - 2\eta_0 + a\eta_1 + \eta_2), \ a \geq 0; \)
2) \( \pm(\eta_{-1} + \eta_0), \)

where

\[
\eta_{-2} = \eta_2 = z_1^2 \bar{z}_2^2, \quad \eta_{-1} = \eta_1 = \frac{1}{2}(z_1 z_2 \bar{z}_2^2 - z_1^2 \bar{z}_1 \bar{z}_2), \quad \eta_0 = \frac{1}{6}(|z_1|^4 - 4|z_1|^2|z_2|^2 + |z_2|^4).
\]

**Proof of the theorems.** According to [1], a biholomorphic transformation \( \Phi \) preserving the origin and the normal form of the equation of \( M \) acts on the polynomial \( F_{220}(z, \bar{z}) \) by means of the \( 2 \times 2 \)-matrix \( C = \lambda e^{i\theta} J^* U \) of its linear part. Here \( \lambda \in \mathbb{R}^+, \ \theta \in [0, 2\pi), \ U \) is a unitary (or a pseudounitary) matrix with determinant equal to 1. \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), the parameter \( \varepsilon \) equals 0 for the pseudoconvex case, and it can take values 0 and 1 if the Levi form is indefinite. We have

\[
F_{220}(z, \bar{z}) = (-1)^{\varepsilon} \lambda^2 F_{220}^*(J^* U z, J^* U \bar{z}). \quad (2)
\]

Therefore, we must study the linear action (2) on the space \( \mathfrak{M}_{22} \) of harmonic polynomials of bidegree \((2, 2)\).

Let us start with the indefinite form case. It is clear that the factor \( e^{i\theta} \) does not affect \( F_{220} \); \( \lambda \) acts on this polynomial as the multiplication by \( \lambda^{-2} \); \( J \) interchanges the variables \( z_1 \) and \( z_2 \) in the polynomial \( F_{220} \) and changes the signs of all the summands in it.

The group \( SU(1, 1) \) coincides (under our choice of the Levi form) with the group \( SL(2, \mathbb{R}) \). Hence, we may consider the standard representation of \( SL(2, \mathbb{R}) \) in the space \( \mathcal{P}_5 = \mathbb{R}^4 \langle X, Y \rangle \) of real homogeneous polynomials of degree 4 in variables \( X, Y \).

This representation of \( SL(2, \mathbb{R}) \) and the original one are coordinate isomorphic by means of the formulas

\[
\sigma_j \mapsto \tilde{\sigma}_j = X^{2-j} Y^{2+j}.
\]

Further, the study of \( SL(2, \mathbb{R}) \)-orbits in the space \( \mathcal{P}_5 \) becomes more convenient if we substitute the polynomials \( \tilde{P} = \sum_{j=0}^4 \omega_{-j} \tilde{\sigma}_j \) in one variable for the elements \( P = \sum \omega_j \tilde{\sigma}_j \) of this space.

The group \( SL(2, \mathbb{R}) \) acts on the roots of the polynomial \( \tilde{P} \) by linear-fractional transformations of the upper half-plane. Putting the real roots of \( \tilde{P} \) (whenever they exist) to zero and infinity it is easy to obtain cases 2)-6) of Theorem 1 depending on the multiplicities of the real roots. If the polynomial has two pairs of complex conjugate roots, then it is possible to move the pair of roots in the upper half plane to the imaginary unit and another purely imaginary number with the positive imaginary part. Then we obtain \( P = \tilde{\sigma}_2 + a\tilde{\sigma}_0 + \tilde{\sigma}_{-2}, \ a \geq 2, \ i.e., \ case \ 1) \) of Theorem 1.

Now let us turn to the pseudoconvex case. Each harmonic polynomial can be written out in the form

\[
F_{220} = a\eta_{-2} + b\eta_{-1} + c\eta_0 + b\eta_1 + \bar{a}\eta_2, \ a, b \in \mathbb{C} \text{ and } c \in \mathbb{R}.
\]

Denote by \( \mathcal{Q}_5 \subset \mathbb{C}^4 \langle X, Y \rangle \) the subspace of homogeneous polynomials \( Q(X, Y) \) of degree 4 in two variables of the form

\[
Q(X, Y) = aX^4 + bX^3 Y + cX^2 Y^2 + bX Y^3 + \bar{a}Y^4 \quad (3)
\]

endowed with the standard \( SU(2) \)-action.

The actions of the groups \( SU(2) \) in the spaces \( \mathfrak{M}_{22} \) and \( \mathfrak{M}_{22} \) expressed in terms of coefficients \( a, b, c \) coincide identically.

Consider the \( SU(2) \)-orbits in the space \( \mathfrak{Q}_5 \). Let \( Q \) be a polynomial of the form (3), and let

\[
\tilde{Q} = a\xi^4 + b\xi^3 + c\xi^2 - \bar{b}\xi + \bar{a}.
\]