It can be easily shown that for any root \( \tau \) of the polynomial \( \widetilde{Q} \) the complex number \( -\tau^{-1} \) also is a root of this polynomial, and it differs from \( \tau \). The group \( SU(2) \) acts on the roots of the polynomial \( \widetilde{Q} \) by means of linear-fractional transformations of the form

\[
\eta = \frac{\alpha \xi + \beta}{-\beta \xi + \alpha}.
\]

Therefore, the polynomial \( \widetilde{Q} \) has a pair of distinct roots that can be transformed by an \( SU(2) \)-action into the pair \( 1, -1 \). The remaining degree of freedom allows one to put the third root on the segment \([0, 1] \subset \mathbb{R} \). As a result, the fourth root occurs in the subset \([-\infty, -1] \) of the real axis. Then the different cases of possible mutual positions of the roots result in the following list of orbit representatives (up to a nonzero real factor):

- a) the third and the fourth roots are \( \tau \) and \( -\tau^{-1} \) with \( 0 < \tau \leq 1 \); here we obtain representatives \( \xi^4 + \gamma \xi^3 - 2\xi^2 - \gamma \xi + 1 \) with \( \gamma = (1/\tau - \tau) \in [0, \infty) \);
- b) the third and the fourth roots are \( 0 \) and \( -\infty \); then the corresponding orbit is represented by the polynomial \( \xi^3 - \xi \).

The dilation \((z, w) \mapsto (\lambda z, \lambda^2 w)\) allows one to make the constant factor above equal to \( \pm 1 \). □

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Complete Convergence of the Pearson Statistics

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The Pearson statistics, also called the chi-square statistics, was introduced by K. Pearson in connection with a number of problems of mathematical statistics. Various applications of this statistics can be found, for example, in the textbook [1]. Here we are interested in its asymptotic behavior. There are many publications devoted to the investigation of asymptotic properties of the Pearson statistics itself and its

numerous modifications (see [1–10] and the literature cited therein). However, it seems that the properties of the Pearson statistics that are true almost everywhere have never been systematically discussed before.

In this note we give sufficient conditions for the complete convergence of the Pearson statistics, and thus, for its convergence with probability 1.

Let a multinomially distributed random vector \( \mathbf{v} = (v_1, \ldots, v_s) \) be defined on a probability space \((\Omega, \mathcal{F}, P)\). According to the definition of the multinomial distribution, for any set of nonnegative integers \( m_1, m_2, \ldots, m_s \) with the total \( n \), we have

\[
P\{v_1 = m_1, v_2 = m_2, \ldots, v_s = m_s\} = \frac{n!}{m_1!m_2! \cdots m_s!} p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s},
\]

where \( p_k, k = 1, 2, \ldots, s \), are positive numbers with unit total.

The Pearson statistics is defined as the random variable

\[
\chi_n^2 = \sum_{k=1}^{s} \frac{n}{p_k} \left( \frac{v_k}{n} - p_k \right)^2.
\]

It is readily seen that the random variable \( v_k \) has the binomial distribution with parameters \( n \) and \( p_k \).

Set \( q_k = 1 - p_k \). By standard calculations we have the following expressions for the expectations, variations, and covarictions of terms in the sum (1):

\[
\begin{align*}
E\left( \frac{n}{p_k} \left( \frac{v_k}{n} - p_k \right)^2 \right) &= q_k, \\
\text{Var}\left( \frac{n}{p_k} \left( \frac{v_k}{n} - p_k \right)^2 \right) &= 2q_k^2 + \frac{q_k}{np_k} (1 - 6p_kq_k), \\
\text{Cov}\left( \frac{n}{p_k} \left( \frac{v_k}{n} - p_k \right)^2, \frac{n}{p_j} \left( \frac{v_j}{n} - p_j \right)^2 \right) &= 2p_kp_j + \frac{1}{n} (2p_k + 2p_j - 6p_kp_j) - \frac{1}{n}, \quad k \neq j.
\end{align*}
\]

Using these formulas, it is easy to find the well-known expressions for the expectation and variation of the Pearson statistics:

\[
E\chi_n^2 = s - 1, \quad \text{Var}\chi_n^2 = 2(s - 1) + \frac{1}{n} \left( \sum_{k=1}^{s} \frac{1}{p_k} - s^2 - 2s + 2 \right).
\]

**Theorem.** If \( s = s_n \) and \( p_k, k = 1, 2, \ldots, s_n \), depend on \( n \) so that

\[
\sum_{n=1}^{\infty} \frac{1}{s_n^2} < \infty, \quad \inf_{n} \left\{ n \min_{1 \leq k \leq s_n} p_k \right\} = \alpha > 0, \quad \sup_{n} \left\{ \frac{s_n}{n} \right\} = \beta < \infty,
\]

then for any \( \varepsilon > 0 \),

\[
\sum_{n=1}^{\infty} P\left\{ \max_{1 \leq m \leq s_n} \left| \sum_{k=1}^{m} \left( \frac{n}{p_k} \left( \frac{v_k}{n} - p_k \right)^2 - q_k \right) \right| \geq \varepsilon s_n \right\} < \infty,
\]

and so

\[
P\left\{ \lim_{n \to \infty} \frac{1}{s_n} \sum_{k=1}^{s_n} \left( \frac{n}{p_k} \left( \frac{v_k}{n} - p_k \right)^2 - q_k \right) = 0 \right\} = 1.
\]

**Proof.** It is readily seen that (3) is a consequence of (2). The estimate (2), in its turn, follows from the inequality

\[
P\left\{ \max_{1 \leq m \leq s_n} \left| \sum_{k=1}^{m} \left( \frac{n}{p_k} \left( \frac{v_k}{n} - p_k \right)^2 - q_k \right) \right| \geq \lambda \right\} \leq \frac{C}{\lambda^2 s_n^2}, \quad \lambda > 0,
\]

where \( C \) is a certain positive constant independent of \( s \) and \( n \).