UNIVERSAL HORN CLASSES AND ANTIVARIETIES
OF ALGEBRAIC SYSTEMS

V. A. Gorbunov and A. V. Kravchenko*

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We define and study universal Horn classes dual to varieties in both the syntactic and the semantic sense. Such classes, which we call antivarieties, appear naturally, e.g., in graph theory and in formal language theory. The basic results are the characterization theorem for antivarieties, the theorem on cores in axiomatizable color-families, and the decidability theorem for universal theories of families of interpretations of formal languages.

In the present article we define and study universal Horn classes that are dual to varieties in both the syntactic and the semantic sense. Such classes — we call them antivarieties — arise naturally, for instance, in graph theory and in formal language theory; see [1]. The basic results of the article are Theorem 1.2 (on a characterization of antivarieties), Theorems 2.4 and 2.8 (on cores in axiomatizable color-families), and Theorem 4.3 (on decidability of universal theories for families of interpretations of formal languages).

1. PROPER UNIVERSAL HORN CLASSES AND ANTIVARIETIES

A sentence in a signature $L$ is said to be universal Horn if it is a conjunction of sentences in the following form:

\( (\forall \overline{x}) (\alpha_1(\overline{x}) \& \ldots \& \alpha_n(\overline{x})) \); \hspace{1cm} (1)

\( (\forall \overline{x}) (\neg \alpha_1(\overline{x}) \lor \ldots \lor \neg \alpha_m(\overline{x})) \); \hspace{1cm} (2)

\( (\forall \overline{x}) (\alpha_1(\overline{x}) \& \ldots \& \alpha_k(\overline{x}) \rightarrow \alpha_{k+1}(\overline{x})) \), \hspace{1cm} (3)

where $\alpha_i(\overline{x})$ are atomic formulas of $L$. A class of $L$-systems is said to be universal Horn if it is a class of models for some set of universal Horn sentences. In turn, sentences such as (1), (2), and (3) are referred to as, respectively, identities, anti-identities, and quasi-identities; classes of systems defined via these and such are called varieties, antivarieties, and quasivarieties.

Since any identity is equivalent to a conjunction of quasi-identities, varieties are quasivarieties. Moreover, every anti-identity on a trivial system is false, but all quasi-identities are true. For an arbitrary universal Horn class $K$, therefore, we have the following two cases:

1. $K$ contains a trivial system $\mathcal{E}_L$;
2. $K$ does not contain $\mathcal{E}_L$ — such classes are called proper.

In the first case, $K$ is a quasivariety; in the second case, adding trivial systems to $K$ also results in a quasivariety, which we denote by $K^+$. (This follows immediately from the Mal'tsev characterization theorem in [2, Sec. 11].)
Thus the study of universal Horn classes reduces to treating quasivarieties. In some cases, as we will see below, it is more convenient and natural to be dealing with arbitrary, in particular, proper, Horn classes. Such naturally arise in various branches of mathematics. Among them, for instance, are graphs without loops, algebraic closure spaces, Cantor algebras, nontrivial rings with unity, and semigroups without idempotents; see [1].

We start by giving an algebraic characterization of proper universal Horn classes; see also [3]. Here, congruences are understood in the sense of [4]. For any class \( K \), denote by \( K^- \) the class of nontrivial systems in \( K \).

**THEOREM 1.1.** For any universal Horn class \( K \) of \( L \)-systems, where \( L \) is a signature with finitely many predicate symbols, the following are equivalent:

1. \( K^- \) is a universal Horn class;
2. the trivial system \( E_L \) is not embeddable in any system of \( K^- \);
3. for any system \( A \in K \), the greatest congruence \( 1_A \) in \( \text{Con}_{K^+} A \) is compact.

**Proof.** That (1) \( \Rightarrow \) (2) is obvious. Conversely, let \( K \) satisfy (2). Then \( K^- \) satisfies the infinite sentence

\[
(\forall x) \left( \bigvee_{G \in L_F} \neg(G(x, \ldots , x) \approx x) \vee \bigvee_{R \in L_P} \neg R(x, \ldots , x) \right),
\]

where \( L_F \) is a set of function symbols, and \( L_P \) a set of predicate symbols, in \( L \). The compactness theorem says that some final part \( \varphi \) of the above sentence is satisfied in \( K^- \). Therefore, \( K^- = K \cap \text{Mod}(\varphi) \), that is, (2) \( \Rightarrow \) (1).

(2) \( \Rightarrow \) (3). Assume that there exists a system \( A \in K \) for which \( 1_A \) is a noncompact congruence in \( \text{Con}_{K^+} A \). Then \( 1_A = \bigcup_{i \in I} \theta_i \) for some chain \( (\theta_i)_{i \in I} \) of congruences \( \theta_i \neq 1_A \) in \( \text{Con}_{K^+} A \). Consequently, \( E_L \cong A/1_A \cong \lim A/\theta_i \). On the other hand, \( \lim A/\theta_i \) belongs to \( K^- \), yielding a contradiction.

(3) \( \Rightarrow \) (2). Suppose that there exists a system \( \tilde{A} \in K^- \) containing a trivial subsystem with universe \( \{ e \} \). Then \( |A| \geq 2 \). In the direct power \( A^\omega \), consider a subsystem \( B \) whose elements are functions assuming the value \( e \) for all but finitely many \( n \in \omega \). Denote by \( p_i \) a projection of \( B \) onto the \( i \)th component and let \( \theta_n = \bigcap_{i \geq n} \text{ker} p_i \) for all \( n \in \omega \). Clearly, \( \{ \theta_n : n \in \omega \} \) is a chain, \( B/\theta_n \cong B \), and \( \theta_n \neq 1_B \) for all \( n \).

Our definitions also imply that \( (\bigvee_{n \in \omega} \theta_n)^0 = 1_B^0 \). On the other hand, since \( B \) contains a trivial subsystem with universe \( \{ e^* \} \), where \( e^*(n) = e \) for all \( n \in \omega \), we have \( B \models R[e^*, \ldots , e^*] \) for any predicate symbol \( R \). Therefore, the factor system \( B/\bigvee_{n \in \omega} \theta_n \) is trivial, and \( 1_B = \bigvee_{n \in \omega} \theta_n \). Thus \( 1_B \) is not a compact element in \( \text{Con}_{K^+} B \), a contradiction.

It is well known that equational logic in a language freed of function symbols is very poor since such, in this instance, contains no other terms but variables. Therefore, equational logic has made great progress for the case of algebras only. One of the goals pursued in this article is to show that the role of varieties for the case of predicate systems is played, in a certain sense, by antivarieties.

For the class \( K \), \( H^{-1}(K) \) denotes a class of all homomorphic preimages of the systems in \( K \), and we write \( V^{-1}(K) \) for the least antivariety containing \( K \). The next theorem is an analog of the Birkhoff HSP-theorem for antivarieties.

**THEOREM 1.2.** For an arbitrary class \( K \), the following conditions are equivalent:

1. \( K \) is an antivariety, i.e., \( K \) is defined by some (possibly empty) set of anti-identities;
2. \( K \) is a universal Horn class and \( H^{-1}(K) \subseteq K \);