UNBOUNDED SEMIDISTRIBUTIVE LATTICES

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We consider two properties which are close to being lower bounded in the class of finite join semidistributive lattices. An example is constructed in which a finite join semidistributive lattice has both these two properties, but it is not lower bounded.

The purpose of this account is to illustrate a construction technique which has proved useful, and apply it to solve an interesting problem. Namely, we answer in the negative the question as to whether a finite lattice satisfying properties (P) and (Q), which are close to boundedness, is bounded. For a more complete discussion of the theory of bounded homomorphisms and lattices, see [1, Ch. II].

1. UNBOUNDED LATTICES

We begin with a well known criterion for meet semidistributivity. If \( L \) is a finite lattice and \( a \in J(L) \), let \( \kappa(a) \) be the largest element above \( a_* \) but not above \( a \), if such an element exists. We regard \( \kappa : J(L) \to M(L) \) as a partial map.

**LEMMA 1.** Let \( L \) be a finite lattice. Then \( L \) satisfies \( SDA \) if and only if \( \kappa(a) \) exists for each \( a \in J(L) \). Moreover, if a finite lattice \( L \) satisfies \( SDA \), then \( \kappa \) maps \( J(L) \) onto \( M(L) \). If \( L \) also satisfies \( SD_v \), then \( \kappa \) is one-to-one, and the dual map \( \kappa^d : M(L) \to J(L) \) is its inverse.

We define the standard dependency relations on \( J(L) \) as follows (assuming \( SDA \) for the first three):

\[
\begin{align*}
A & \leq B \iff b < a < \kappa(b) \gamma; \\
A & \leq B \iff a \neq b, b \not\leq \kappa(a), \text{ and } b_* \leq \kappa(a); \\
A & \leq b \iff a \mathrel{\kappa} a, \text{ and } b \mathrel{\kappa} b; \\
A & \leq B \iff \text{there exists } x \in L \text{ such that } a \leq b \oplus x \text{ but } a \not\leq b \oplus x.
\end{align*}
\]

Thus \( A \cup B = C \subseteq D \).

The dual relations are defined on \( M(L) \). In semidistributive lattices, they behave particularly nicely, as is shown by the following result.

**LEMMA 2** (see [2]). Let \( L \) be a finite semidistributive lattice and let \( a, b \in J(L) \). We have:

1. \( A \leq B \) if and only if \( \kappa(a) \mathrel{\kappa} B \).
2. \( A \leq B \) if and only if \( \kappa(a) \mathrel{\kappa} B \).

Recall the basic results on boundedness (in the sense of McKenzie) and \( D \)-cycles.

**THEOREM 3.** A finite lattice \( L \) is lower bounded if and only if \( L \) contains no \( D \)-cycle. Moreover, every lower bounded lattice satisfies \( SD_v \).

**THEOREM 4.** A finite semidistributive lattice \( L \) is bounded if and only if \( L \) contains no \( C \)-cycle.
We will be interested in semidistributive lattices which satisfy the condition

\( (P) \) if \( a, b \in J(L) \) and \( b < a \), then \( aAb \)

and its dual

\( (Q) \) if \( p, q \in M(L) \) and \( q > p \), then \( pAdq \).

Note that \( pAdq \) is equivalent to \( \kappa^d(p)B\kappa^d(q) \).

In [3], Caspard proved that the lattice \( S_n \) of all permutations of an \( n \)-element set is bounded. In [4], she proved that these lattices satisfy \( (P) \) and \( (Q) \), and used this to give a nice characterization of the linear orders on \( J(S_n) \) which are consistent with the dependency relation \( D \). Thus it seems natural to ask the following:

If a finite semidistributive lattice satisfies \( (P) \) and \( (Q) \), must it be bounded?

We will show that the answer is ‘no.’

2. SEMIDISTRIBUTIVE LATTICES

**Lemma 5.** If \( L \) is a finite lattice which fails SD\( _\lor \), then there exist distinct elements \( a, b \in J(L) \) and \( c \in L \) such that \( a \lor c = b \lor c > c \), \( a_* = c \), and \( b_* = c \).

**Proof.** Suppose \( a_0 \lor c_0 = b_0 \lor c_0 > (a_0 \land b_0) \lor c_0 \) in \( L \). Choose \( c \) such that \( a_0 \lor b_0 > (a_0 \land b_0) \lor c_0 \). Then choose \( a \) minimal such that \( a \leq a_0 \), but \( a \not\leq c \), and choose \( b \) minimal such that \( b \leq b_0 \) but \( b \not\leq c \).

As an immediate application, we have the following results, which characterize join semidistributivity as a sort of weak lower boundedness.

**Theorem 6.** Let \( L \) be a finite lattice. Then \( L \) fails SD\( _\lor \) if and only if there exist distinct elements \( a, b \in J(L) \) and \( x \in L \) such that \( a \lor x = b \lor x \), \( a \not\leq b \lor x \), and \( b \not\leq a \lor x \).

**Corollary 7.** If \( L \) is a finite lattice which fails SD\( _\lor \), then \( L \) is not lower bounded. In fact, there exist \( a, b \in J(L) \) such that \( aDbDa \) via the same element \( x \).

Figure 1 gives a lattice which satisfies SD\( _\lor \) but which contains a short cycle \( aDbDa \), via distinct elements \( x \) and \( y \).

**Theorem 8.** Let \( L \) be a finite lattice which satisfies SD\( _\land \). Then \( L \) fails SD\( _\lor \) if and only if there exist \( a, b \in J(L) \) such that \( aBbBa \).